Introduction to Matsumoto metric

H. Shimada, S. V. Sabau[a]

[a]Hokkaido Tokai University,
5-1-1-1 Minamisawa, Minami-ku, Sapporo, 005-8601 Japan

Even though the notion of Matsumoto metric is a quite old topic, very few people took a close look into its true nature. In general, it is thought as a particular case of the class of (α, β)-metrics with the variables α and β being independent. This is not exactly the case, as we shall see.

In the present paper, we are going to reconsider the original meaning of the Matsumoto metric and to study some of its fundamental properties from a new perspective.

1. The indicatrix of the Matsumoto metric is a Limaçon

It can be easily seen that if a person walks on a plane with an angle ε of inclination in a direction θ with constant speed $v(:= a)$, then the distance walked in the time $t = 1$ is

$$r = a + w \cos\theta, \tag{1}$$

where $w = g/2 \sin\varepsilon$, g is the gravity constant, and (r, θ) are the polar coordinates in plane (see [1]).

Equation (1) is the equation of a limaçon, and from a result of Pascal, the limaçon is convex if and only if

$$a \geq 2w.$$

2. How is Matsomoto metric induced?

Let us consider a surface S embedded in the usual Euclidean space $\mathbb{R}^3$, i.e.

$$S \hookrightarrow \mathbb{R}^3, \quad (x, y) \in S \mapsto (x, y, z = f(x, y)) \in \mathbb{R}^3.$$

It is known that the induced Riemannian metric on the surface S is given by

$$(a_{ij}) = \begin{pmatrix} 1 + (f_x)^2 & f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix}$$

where f_x and f_y means partial derivative with respect to x and y, respectively.

If we consider now a coordinate system $(x, y, u, v) \in TM$ in the tangent bundle TM, then the limaçon equation (1) becomes

$$u^2 + v^2 + \beta^2 = a\sqrt{u^2 + v^2 + \beta^2} - w\beta, \tag{2}$$

where we have put

$$\beta = f_x u + f_y v = df, \quad w = \frac{g}{2}.$$

Since the left hand side of (2) is exactly the induced Riemannian metric, we put further

$$\alpha^2 = a\alpha - w\beta. \tag{3}$$

In other words,

$$\begin{aligned}\alpha^2 &= u^2 + v^2 + (f_x u + f_y v)^2 \\ &= (1 + f_x^2)u^2 + 2f_x f_y uv + (1 + f_y^2)v^2\end{aligned}$$

By means of K. Okubo method, i.e. "*If* $h(v) = 0$ *is the indicatrix of the Minkowski space* V^n*,where* $h := F - 1$*, then the fundamental function* F *is derived from* $h(\frac{v}{F}) = 0$*.*", from (3) one gets

$$F = \frac{\alpha^2}{a\alpha - w\beta}.$$

By normalization, it results

$$F = \frac{\alpha^2}{\alpha - \beta}. \tag{4}$$

This is the so called **Matsumoto metric**([3]).

But is this a Finsler metric? In other words, it satisfies the following axioms:

1. $F > 0$, i.e. $\alpha - \beta > 0$
2. F is homogeneous of degree 1 with respect to (u, v),
3. Is the fundamental tensor g_{ij} positive definite, or equivalently, is the indicatrix convex?

Theorem 1.

A 2-dimensional differential manifold M *endowed with the fundamental function (4) is a Finsler manifold if and only if* $f_x^2 + f_y^2 \leq \frac{1}{3}$*.*

Proof.

The first two axioms are easily checked. As for the convexity of the indocatrix, the condition $a \geq 2w$ can be rewritten as $\frac{1}{2}g \geq g \sin\varepsilon$. Here

$$\sin\varepsilon = \frac{\sqrt{f_x^2 + f_y^2}}{\sqrt{1 + f_x^2 + f_y^2}}$$

([1]).

It follows,

$$(\frac{1}{2})^2 \geq \sin^2 \varepsilon = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}.$$

And from here one obtains

$$f_x^2 + f_y^2 \leq \frac{1}{3}.$$

The inverse is also true.

QED

Example.

Let us consider the surface S to be a plane, say $z = ax + by + c$. The convexity condition for the indicatrix reads now

$$a^2 + b^2 \leq \frac{1}{3}.$$

Remark.

If $df = 0$ the Matsumoto metric reduces to a Riemannian metric.

Proposition 2.

The 1-form β is parallel, i.e. $b_{i|j} = 0$, if and only if $f_{xx} = f_{xy} = f_{yy} = 0$, or equivalently, $f(x, y) = ax + by + c$ is a plane. Here "$|$" represents the covariant derivative with respect to the Riemannian connection of (a_{ij}), and $a,\ b,\ c$ are constants.

Proof.

The Christoffel symbols of the Riemannian metric (a_{ij}) are

$$\gamma_{11}^1 = \frac{f_x}{1 + f_x^2 + f_y^2} f_{xx}, \quad \gamma_{11}^2 = \frac{f_y}{1 + f_x^2 + f_y^2} f_{xx}$$

$$\gamma_{12}^2 = \frac{f_y}{1 + f_x^2 + f_y^2} f_{xy}, \quad \gamma_{22}^2 = \frac{f_y}{1 + f_x^2 + f_y^2} f_{yy}$$

$$\gamma_{12}^1 = \frac{f_x}{1 + f_x^2 + f_y^2} f_{xy}.$$

Using these, one gets

$$b_{1|1} = \frac{1}{1 + f_x^2 + f_y^2} f_{xx}$$

$$b_{1|2} = b_{2|1} = \frac{1}{1 + f_x^2 + f_y^2} f_{xy}$$

$$b_{2|2} = \frac{1}{1 + f_x^2 + f_y^2} f_{yy}.$$

Therefore we obtain the conclusion.

QED

We recall here from [4]

Theorem 3.(M. Matsumoto)

For a Finsler manifold with (α, β)-metric, $b_{i|j} = 0$ if and only if $D^i_{jk} = 0$, where $D^i_{jk} = G^i_{jk} - \gamma^i_{jk}$. Here "$|$" is the covariant derivative with respect to the Riemannian metric α, γ^i_{jk} its Christoffel coefficients, and G^i_{jk} the coefficients of Berwald connection of the given Finsler manifold with (α, β)-metric.

Since the Matsumoto metric is a special case of 2-dimensional Finsler manifold with (α, β)-metric, from here and Proposition 2 we obtain

Theorem 4.

For a Matsumoto metric on the surface S, if $b_{i|j} = 0$, then the Matsumoto metric is a Berwald one and the surface S has to be a plain.

Remark.

A Berwald space is a Finsler space where the coefficients of Berwald connection $G^i_{jk} = G^i_{jk}(x)$.

Remark.

The rigidity theorem of Z. Szabo for Finsler surfaces states that on Berwald surface is a locally Minkowski space or a Riemannian surface depending on the Gauss curvature of the surface ([2]). In our case, since $b_{i|j} = 0$ implies $f(x, y) = ax + by + c$, with a, b, c constants, it follows

$$(a_{ij}) = \begin{pmatrix} 1 + a^2 & ab \\ ab & 1 + b^2 \end{pmatrix}, \quad \beta = df = au + bv.$$

Therefore, a Matsumoto surface that is Berwald it has to be in fact a Minkowski metric in the plain with Gauss curvature $K = 0$.

REFERENCES

1. P. L. Antonelli, R. Ingarden, M. Matsumoto, The theory of sprays and Finsler spaces with applications in physics and biology, Kluwer Acad. Publ., FTPH vol. 58, 1993.
2. D. Bao, S. S. Chern, Z. Shen, An introduction to Riemann–Finsler geometry, Springer, GTM 200, 2000.
3. M. Matsumoto, A slope of a mountain is a Finsler surface with respect to a time measure, J. Math. Kyoto Univ. 29 (1989), 17.
4. M. Matsumoto, Theory of Finsler spaces with (α, β)-metrics, Rep. on Math. Phys., 31 (1991), 43.