PROJECTIVELY FLAT FINSLER METRICS CORRESPONDING TO QUADRICS IN $\mathbb{C}P^n$

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Introduction

We are going to study Finsler structures on projectively flat spheres. The problem is studied for the first time by Bryant in [Br 2] for two dimensional spheres. We generalize his approch to the n-dimensional spheres $S^n \subset \mathbb{R}^{n+1}$.

The method is in essence the same as Okubo method, i.e. starting from the indicatrix find the fundamental function F.

Let us start with a Finsler space in which the geodesic foliation is amenable, i.e. the space of geodesics is a Hausdorff smooth manifold. Then the given Finsler structure induces on the manifold of geodesics the structure of a symplectic Riemannian manifold.

Conversely, starting with a manifold (that mimics the manifold of geodesics), then one can recover the Finsler structure. The new Finsler structure will depend on some parameters being in fact a family of Finsler structures that includes the initial one.

In §1 we construct the tangent space of the projective sphere $\mathbb{S}$ and show that S^n and $\mathbb{S}$ are isomorphic. This allows to study $\mathbb{S}$ insteand of S^n. Using a theorem proved on the two dimensional case by Funk and in arbitrary dimension by Busemann we show that the 'projectivized' Grassmannian $\mathbb{G}$ of 2-planes in $\mathbb{R}^{n+1}$ is the Geodesic space of the a Finsler manifold $(\mathbb{S}, F)$ of constant flag curvature $K = 1$, §2. The inverse image of $\mathbb{G}$ in $\mathbb{CP}^n$ is a holomorphic quadric C that allows to find explicitly the fundamental function F on $\mathbb{S}$ (§3, §4).

§1. The Projective Space $\mathbb{S}$.

We consider the canonical vector space R^{n+1}, and construct the *projective sphere*

$$\mathbb{S} = (R^{n+1} \setminus \{0\})/R^+. \tag{1.1}$$

Let us consider also the usual n-sphere $S^n = \{u \in R^{n+1} : ||u|| = 1\}$, where $||.||$ is the norm induced by the usual inner product $\cdot$ in R^{n+1}. One can see that an element $[v] \in \mathbb{S}$ is a set $\{\lambda \cdot v : \lambda \in R^+, v \in R^{n+1}\}$.

Then there is a diffeomorphism

$$\begin{aligned} &\psi : \mathbb{S} = (R^{n+1} \setminus \{0\})/R^+ \to S^n = \{u \in R^{n+1} : ||u|| = 1\}, \\ &\psi[v] = \frac{v}{||v||}, \end{aligned} \tag{1.2}$$

with the inverse

$$\psi^{-1} : S^n = \{u \in R^{n+1} : ||u|| = 1\} \to \mathbb{S} = (R^{n+1} \setminus \{0\})/R^+. \\ \psi^{-1}(v) = [v]\ . \tag{1.3}$$

It can be checked easily that the above maps are independent of the choice of representative.

We are going to construct the differential of ψ in (1.2). First we need the tangent bundles of S^n and $\mathbb{S}$, namely TS^n, and $T\mathbb{S}$, respectively.

One can see that $TS^n = \{(v, w) \in R^{n+1} \times R^{n+1} : ||v|| = 1,\ v \cdot w = 0\}$.

Indeed, let us consider a point $v \in S^n \subset R^{n+1}$, and let us denote $v^\perp := \{w \in R^{n+1} : v \cdot w = 0\}$. We are going to show that $v^\perp$ is isomorphic to T_vS^n, for any $v \in S^n$. First, we consider the inclusion map $f : v \in S^n \to f(v) = v \in R^{n+1}$, and let $\gamma : [-\varepsilon, \varepsilon] \to S^n$ be a smooth curve on S^n. If we write $v \in S^n$ for the point on γ such that $\gamma(0) = v$, then we can define a tangent vector to the manifold S^n at the point v by:

$$X_v = \frac{d}{dt}\gamma(t)|_{t=0} \in T_vS^n,$$

and any tangent vector to S^n arises from some γ.

The tangent map of the canonical inclusion f reads

$$df_v : T_vS^n \to T_{f(v)}R^{n+1}, \quad X_v \to df_v(X_v) = \frac{d}{dt}f(\gamma(t))|_{t=0}. \tag{1.4}$$

On the other hand, $||f(\gamma(t))|| = 1$, i.e. $f(\gamma(t)) \cdot f(\gamma(t)) = 1$. Taking now the derivative with respect to t of this relation we obtain that $f(\gamma(t))$ and $\frac{d}{dt}\gamma(t)$ are perpendicular, and in particular for $t = 0$, we obtain that T_vS^n and $v^\perp$ are isomorphic by df_v.

To construct $T\mathbb{S}$ is a little more complicated.

For any pair of vectors $\mathbf{v} = (v, w)$, $v, w \in R^{n+1}$, $v \neq 0$, we consider the curve on $\mathbb{S}$ defined by the formula

$$\gamma_{\mathbf{v}} : (-a, a) \to \mathbb{S}, \qquad \gamma_{\mathbf{v}}(t) = [v + tw]. \tag{1.5}$$

The velocity vector at $t = 0$ of this curve is $\frac{d}{dt}[v + tw]|_{t=0}$ and we are going to use it to construct an equivalence relation, denoted by "$\approx$" , on $R^{n+1} \setminus \{0\} \times R^{n+1}$ in the following way: for any 2 pairs $(v_1, w_1),\ (v_2, w_2) \in R^{n+1} \setminus \{0\} \times R^{n+1}$, we define $(v_1, w_1) \approx (v_2, w_2)$ if $[v_1] = [v_2]$ and $\frac{d}{dt}[v_1 + tw_1]|_{t=0} = \frac{d}{dt}[v_2 + tw_2]|_{t=0}$. We will denote the equivalence classes of (v, w) with respect to this equivalence relation by $[v, w]$.

Proposition 1.1. *(i) If $(v_1, w_1),\ (v_2, w_2) \in R^{n+1} \setminus \{0\} \times R^{n+1}$, then $(v_1, w_1) \approx (v_2, w_2)$ if $(v_1, w_1) = (av_2, aw_2 + bv_2)$, for some $a, b \in R$, $a > 0$.*

(ii) If $(v, w) \in R^{n+1} \setminus \{0\} \times R^{n+1}$, then the relation $[v, w] = [av, aw + bv]$ holds for all real numbers $a > 0$, and b.

Proof.

To prove (i), let us remark first that because $\Psi : \mathbb{S} \to S^n$ is a diffeomorphism (see (1.1)), $\frac{d}{dt}[v_1 + tw_1]|_{t=0} = \frac{d}{dt}[v_2 + tw_2]|_{t=0}$ is equivalent to $\frac{d}{dt}(\psi[v_1 + tw_1]|_{t=0}) = \frac{d}{dt}(\psi[v_2 + tw_2]|_{t=0})$, and by an elementary calculation one can see that

$$\frac{d}{dt}\left(\frac{v + wt}{||v + wt||}\right)|_{t=0} = \frac{w}{||v||} - \frac{v \cdot w}{||v||^3}v. \tag{1.6}$$

This is an important result to be used in the following.

Now, if we put $a := \frac{1}{||v_2||}$, and $b = -\frac{v_2 \cdot w_2}{||v||^3}$, one can check by straightforward calculation using (1.1) that $\frac{d}{dt}(\psi[v_1 + tw_1]|_{t=0}) = \frac{d}{dt}(\psi[v_2 + tw_2]|_{t=0})$, i.e. $\frac{d}{dt}[v_1 + tw_1]|_{t=0} = \frac{d}{dt}[v_2 + tw_2]|_{t=0}$ so the statement (i) is proved.

Let us consider the curves $\gamma_{\mathbf{v}}, \delta_{\mathbf{v}} : (-a, a) \to \mathbb{S}$, where $\gamma_{\mathbf{v}}$ is given in (1.5), and $\delta_{\mathbf{v}}$ is defined by $\delta_{\mathbf{v}} = [av + t(aw + bv)]$, for all real numbers $a > 0$, and b. Now all we have to do is to prove that

$$\frac{d}{dt}[v + tw]|_{t=0} = \frac{d}{dt}[av + t(aw + bv)]|_{t=0} \tag{1.6'}$$

because this means $[v, w] = [av, aw + bv]$. From (1.6) we have:

$$\frac{d}{dt}[av + t(aw + bv)]|_{t=0} = \frac{d}{dt}\frac{av + t(aw + bv)}{||av + t(aw + bv)||}|_{t=0} = \frac{aw + bv}{||av||} - \frac{av(aw + bv)}{a^3||v||^3}av.$$

A straightforward computation shows that (1.6)' is true. $\square$

One can see that the identity $c[v, w] = [v, cw]$ holds good for any $c \in R$.

We are going to show that $[v, w]$ represents an element of the vector space $T_{[v]}\mathbb{S}$.

Proposition 1.2. *The quotient space $(R^{n+1} \setminus \{0\} \times R^{n+1})/_{\approx}$ is isomorphic to $T\mathbb{S}$.*

Indeed, taking again into account the fact that $\psi : \mathbb{S} \to S^n$, defined in (1.2), is a diffeomorphism, its tangent map $d\psi : T\mathbb{S} \to TS^2$ is also a diffeomorphism, i.e. we have

$$d\psi^{-1}(TS^n) = T\mathbb{S}. \tag{1.7}$$

On the other side, we can construct a map $\Psi : (R^{n+1} \setminus \{0\} \times R^{n+1})/_{\approx} \to TS^n$, defined by

$$\Psi([v, w]) = \left(\frac{v}{||v||}, \frac{w}{||v||} - \frac{v \cdot w}{||v||^3}v\right), \tag{1.8}$$

for any $[v, w] \in (R^{n+1} \setminus \{0\} \times R^{n+1})/ \approx$. The inverse is given by

$$\Psi^{-1}(v, w) = [v, w], \tag{1.9}$$

for any $(v, w) \in TS^n$.

One can see that Ψ is a diffeomorphism and moreover, $\Psi = d\psi$. Therefore we have $d\psi^{-1}(TS^n) = (R^{n+1} \setminus \{0\} \times R^{n+1})/_{\approx}$, and taking into account (1.7) the proposition is proved.

Moreover we have constructed the diffeomorphism $d\psi = \Psi : T\mathbb{S} \to TS^n$, given by (1.8), with the inverse given in (1.9).

§2. Projectively parametrized lines in $\mathbb{S}$.

Let us consider the the vector space $V = R^{n+1}$, and $\mathbb{S}$ the projective sphere as above.

Let us start with the manifold $M = S^n$ which geometry was already depicted above. Its manifold of geodesics is the Grassmannian manifold $G_2^+(\mathbb{R}^{n+1})$ of oriented two-planes in $\mathbb{R}^{n+1}$, provided the paths on S^n to be the 'great circles'.

We are interested in the path geometry of a Finsler structure of constant flag curvature on $\mathbb{S}$. In general is very dificult to estimate the existence of the manifold of geodesics of it. However, if we ask for the given Finsler structure to be projectively flat, i.e. to have the standard geodesics, then we know that its Manifold of Geodesics is $G_2^+(\mathbb{R}^{n+1})$.

Hence, denoting by Σ the indicatrix of a projectively flat Finsler structure of constant flag curvature $K = 1$ on S^n, then Σ carries a structure of generalized path geometry.

However, we would like to projectivize considering $M = \mathbb{S}$, the projective n-sphere. Therefore we replace S^n by $\mathbb{S}$, considering the path geometry of $\mathbb{S}$.

The generalized path geometry defined on Σ decends to a well defined (classical) path geometry on $\mathbb{S}$. This is, of course, the classical geometry, where the paths of $\mathbb{S}$ are the 'great circles'.

In order to realize this path geometry, let us consider an oriented 2-dimensional subspace $P \subset R^{n+1}$, i.e. a 2-plane, and let $\mathbf{v} = (v_0, v_1)$ be an oriented basis of P, i.e. $P = span\{v_0, v_1\}$.

Any vector $w \in P$ can be written in the form $w = \lambda(cos\ s\ v_0 + sin\ s\ v_1)$ for some $s \in [0, 2\pi], \lambda \in R^+$, i.e.

$$P = \{\lambda(cos\ s\ v_0 + sin\ s\ v_1)|\lambda \in R^+,\ s \in R,\ \mathbf{v} = (v_0, v_1)\ fixed\ basis\}. \tag{2.1}$$

A path of the considered path geometry on $\mathbb{S}$ is the oriented line $[P]$ in $\mathbb{S}$, defined as the oriented curve parametrized by the map $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$ defined by the formula

$$\gamma_{\mathbf{v}}(s) = [cos\ s\ v_0 + sin\ s\ v_1] \tag{2.2}$$

together with the convention that S^1 be oriented so that ds is a positive 1-form on S^1.

The set

$$[P] = \{[v] \in \mathbb{S} | v \in P \setminus \{0\}\}.$$

can be written as

$$[P] = \{\gamma_{\mathbf{v}}(s) | s \in [0, 2\pi]\} = \{[cos\ s\ v_0 + sin\ s\ v_1] | s \in [0, 2\pi]\} \subset \mathbb{S}.$$

Since P is independent on the choice of basis $\mathbf{v}$, the set $[P]$ is also independent of $\mathbf{v}$.

And since $[P]$ is not dependent on the basis, taking for example an orthogonal basis, it follows that $[P]$ is indeed a great circle of the projective sphere $\mathbb{S}$.

An open domain $\mathbb{D} \subset \mathbb{S}$ will be said to be *convex* if its intersection with each line in $\mathbb{S}$ is connected.

If $\mathbb{D}$ is convex, then we denote by $\mathbb{D}^*$ the set of oriented lines in $\mathbb{S}$ whose intersection with $\mathbb{D}$ is non-empty.

Definition 2.1.
A Finsler structure $\Sigma_{\mathbb{D}} \subset T\mathbb{D}$, will be said to have linear geodesics if each of its (oriented) geodesics is of the form $[P] \cap \mathbb{D}$ for some (unique) oriented 2-plane P.

The linear geodesics of the canonical structure on $\mathbb{S}$ are the paths $[P]$ (the paths of the classical geometry of $\mathbb{S}$).

Lemma 2.1.
The tangent vector to the curve $\gamma_{\mathbf{v}}$ is given by

$$\gamma'_{\mathbf{v}}(s) = [-sin\ s\ v_0 + cos\ s\ v_1]. \tag{2.3}$$

(The proof is just a simple calculation.)

One can remark that in general, the vector $\gamma'_{\mathbf{v}}$ is not unit length, since the oriented basis $\mathbf{v}$ is an arbitrary one.

We can now introduce an orientation on the line $[P]$. In order to do this is enough to consider an orientation in P, i.e. to fix an order for the basis $\mathbf{v}$, say (v_0, v_1), and a positive orientation on the real numbers R. It is obvious that this orientation depends on the chosen basis $\mathbf{v}$.

The choice of oriented basis $\mathbf{v}$ of $[P]$ affects this parametrization but any two such choices will yield the same image line with same orientation.

The next step is to construct the *induced metric* on $[P]$. We saw above that different choices of the basis $\mathbf{v}$ induce different parametrizations. Indeed, if we choose the basis $\mathbf{v} = (v_0, v_1)$ and $\mathbf{w} = (w_0, w_1)$ we have the parametrizations $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$, and $\gamma_{\mathbf{w}} : S^1 \to \mathbb{S}$, respectively, so that $Im\ \gamma_{\mathbf{v}} = Im\ \gamma_{\mathbf{w}}$.

Therefore, in general, we can consider a diffeomorphism $\alpha : S^1 \to S^1$, given by $\alpha = \alpha(s)$, such that $\gamma_{\mathbf{v}}(s) = \gamma_{\mathbf{w}}(\alpha(s))$. If we fix a "standard metric" ds^2 on the curve $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$, then the induced metric on $[P]$ is $d\alpha^2$.

Moreover, let us consider as above $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$, and $\gamma_{\mathbf{w}} : S^1 \to \mathbb{S}$, with the corresponding metrics ds^2, and $d\alpha^2$, respectively. These two parametrizations have the same orientation and same induced metric iff there is a real number s_0 such that $\alpha(s) = s - s_0$.

Theorem 2.2. *[Funk, Busemann]*
Let $\Sigma_{\mathbb{D}} \subset \mathbb{S}$ be a convex domain in $\mathbb{S}$ and suppose that there is a Finsler structure $\Sigma_{\mathbb{D}}$ on $\mathbb{D}$ with linear geodesics and whose curvature satisfies K=1.
Then, for every oriented line $[P] \in \mathbb{D}^*$, there exists an oriented basis $\mathbf{v} = (v_0, v_1)$ of [P] so that the parametrization $\gamma_{\mathbf{v}}$ has unit speed (i.e. is a $\Sigma_{\mathbb{D}}$-curve).

The Theorem of Funk-Busemann tell us that in the case of a Finsler structure $\Sigma_{\mathbb{D}}$ on $\mathbb{D}$ with linear geodesics and whose curvature satisfies $K = 1$, among the many bases of P there is at least a distinguished one, let us denote it by $\widetilde{\mathbf{v}}$. It has the property that the geodesic $\gamma_{\widetilde{\mathbf{v}}}$ is a unit speed geodesic with respect to the given Finsler metric on $\mathbb{D}$, i.e. if we denote the fundamental function of the Finsler structure $\Sigma_{\mathbb{D}}$ on $\mathbb{D}$ by $\nu : T\mathbb{D} \to R^+$, then $\nu(\gamma_{\widetilde{\mathbf{v}}}) = 1$.

On the other hand, a Finsler structure on a manifold induces a metric on the curves on the manifold. Using the above notation, a metric determined by the Finslerian structure ν on the curve $\gamma_{\widetilde{\mathbf{v}}}(t)$ is $ds^2 = \nu(\gamma_{\widetilde{\mathbf{v}}})^2 dt^2$. This metric depends essentially on the Finsler metric and on the chosen basis $\widetilde{\mathbf{v}}$. If we choose the distinguished basis $\widetilde{\mathbf{v}}$ given by Funk-Busemann's theorem, then the curve is natural parametrized.

Let $\widetilde{\mathbf{v}} = (v, w)$ be the oriented basis with respect to which the linear geodesic is unit speed. For an arbitrary basis $\mathbf{v} = (v_0, v_1)$ we put

$$v = v_0,$$
$$w = a_0 v_0 + b_0 v_1,$$

where a_0, b_0 are real numbers. Then by a simple calculation:

$$\gamma'_{\widetilde{\mathbf{v}}} = [v, -sin\ s\ v + cos\ s\ w] = [v_0, av_0 + bv_1] \in \Sigma,$$

with $a, b \in R, b \neq 0$.

Moreover,

$$[v_0, av_0 + bv_1] = [v_0, b(v_1 + \frac{a}{b}v_0)] = b[v_0, v_1 + \frac{a}{b}v_0] = b[v_0, v_1] \in \Sigma, \tag{2.4}$$

for an arbitrary oriented basis $\mathbf{v} = (v_0, v_1)$.

Let us denote by $\mathbb{G}$ the space of oriented great circles in $\mathbb{S}$. Namely, a point in $\mathbb{G}$ will be $[P] = [v_0 \wedge v_1]$, where $V = (v_0, v_1)$ is an oriented basis of P. This is the 'projectivized' Grassmannian of 2-planes in $\mathbb{R}^{n+1}$.

The resulting oriented line $[v_0 \wedge v_1] \in \mathbb{G}$ depends only on the oriented plane P, not on the choise of oriented basis V.

Indeed, if we consider another basis, say $\mathbf{w} = (w_0, w_1)$ such that $w_0 = av_0 + bv_1$, $w_1 = cv_0 + dv_1$, with $a, b, c, d \in R$, then $w_0 \wedge w_1 = (av_0 + bv_1) \wedge (cv_0 + dv_1) = (ad - bc) v_0 \wedge v_1$. Here, because of the orientation of the basis, we have $\lambda = ad - bc > 0$. Therefore, $\alpha_{\mathbf{w}}(v) = w_0 \wedge w_1 \wedge v = \lambda v_0 \wedge v_1 \wedge v$.

Remark: In the 2-dimensional case, the manifold of oriented geodesics $\mathbb{G}$ coincides with the space $\mathbb{S}^* = V \setminus \{0\}/\mathbb{R}^+$.

Moreover, the points of $\mathbb{S}^*$ can be found in the following way.

We fix a standard volume form on V, i.e. the standard identification of $\Lambda^3(V)$ with R. Thus, for any three vectors $v_0, v_1, v_2 \in V$, the wedge product $v_0 \wedge v_1 \wedge v_2$ will be treated as a number and this identification is invariant under the natural action of $SL(3, R)$ on V.

Indeed, the volum form of the 3-dimensional Riemannian manifold (M, g) means a 3-form, say $dv \in \Lambda^3(M)$, such that $dv = \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3$.

Then, for every 3-form, say $\omega \in \Lambda^3(V)$, there exists a real number $c \in R$, depending on ω, i.e. $c = c(\omega)$, such that $\omega = c(\omega) dv$. For any three vectors $v_0, v_1, v_2 \in V$, the wedge product $v_0 \wedge v_1 \wedge v_2$ belongs to $\Lambda^3(V)$. Therefore for this 3-form we can write $v_0 \wedge v_1 \wedge v_2 = c(v_0, v_1, v_2) dv$. And so we can identify $v_0 \wedge v_1 \wedge v_2$ with its "coefficient" $c(v_0, v_1, v_2) \in R$. Therefore we can construct the following identification:

$$\phi : \Lambda^3(V) \to R, \phi(\omega) = c(v_0, v_1, v_2).$$

If we consider now the natural action of $SL(3, R)$ on V, given by $SL(3, R) \times V \to V$ given by $(g, v) \mapsto g \cdot v$, then we can define $A \cdot v_0 \wedge v_1 \wedge v_2 = Av_0 \wedge Av_1 \wedge Av_2 = det\ Av_0 \wedge v_1 \wedge v_2$ by straightforward calculation. On the other hand, $A \in SL(3, R)$, i.e. $det\ A = 1$, therefore the identification $v_0 \wedge v_1 \wedge v_2 \mapsto c(v_0, v_1, v_2)$ is invariant under this action.

A choice of oriented basis $\mathbf{v}$ also defines a linear functional $\alpha_{\mathbf{v}} : V \to R$ by the rule $\alpha_{\mathbf{v}}(v) = v_0 \wedge v_1 \wedge v$, where $\alpha_{\mathbf{v}}(v) = c(v_0, v_1, v) \in R$.

The resulting oriented line $[\alpha_{\mathbf{v}}] \in \mathbb{S}^*$ depends only on the oriented plane P, not on the choice of oriented basis $\mathbf{v}$, so we can identify the points of $\mathbb{S}^*$ with the space of oriented lines in $\mathbb{S}$ via the identification $[E] = [\alpha_{\mathbf{v}}]$.

Therefore we will think of $\mathbb{G}$ as the manifold of linear geodesics of $\mathbb{S}$.

From now on, a point of $\mathbb{G}$ will be $[P] = [v_0 \wedge v_1] \in \mathbb{G}$.

In arbitrary dimensions, the path geometry of a projective flat Finsler structure of a constant flag curvature $K = 1$ on $\mathbb{S}$ will be given by a double fibration

(2.5)
$$\begin{array}{ccc} & \Sigma & \\ \swarrow^{\lambda} & & \searrow^{\tau} \\ \mathbb{G} & & \mathbb{S} \end{array}$$

where the projections, $\tau : \Sigma \to \mathbb{S}$ and $\lambda : \Sigma \to \mathbb{G}$ are smooth submersions.

From the general theory we know that the manifold of geodesics has the structure of a Riemannian manifold. Let us see how is this structure is obtained.

Since the integral curves of $\Sigma = X_1$, i.e. the geodesics of the Finsler structure, are periodic of period 2π on Σ, it follows that that we have a free action of the unit circle on Σ whose orbits are the leaves of $\mathcal{P}$, i.e. the fibres of the submersion $\lambda : \Sigma \to \mathbb{G}$. Thus, Σ can be regarded as a principal S^1-bundle over $\mathbb{G}$.

However, to study the manifold $\mathbb{G}$ using its Riemannian structure is quite difficult. This is the reason that different authors uses methods from algebraic geometry to do it, [Br], [Fn].

We saw that on any oriented line [P] we have a metric and an orientation induced by the oriented basis $\mathbf{v}$. The following question is natural: *when two oriented basis of P induce the same metric and orientation in the line [P]?*

The question will be considered first in the most general case, i.e. no a priori given metric structure on the base manifold will be assumed.

Theorem 2.3. *Two oriented basis $\mathbf{v} = (v_0, v_1)$ and $\mathbf{w} = (w_0, w_1)$ of P induce the same metric and orientation in the line [P] if and only if there exists a constant s_0 so that*

(2.6)
$$v_0 + iv_1 = re^{is_0}(w_0 + iw_1), \quad \textit{where } i = \sqrt{-1}.$$

Proof.

Let us denote the metrics on the lines $\gamma_{\tilde{\mathbf{v}}}(s)$ and $\gamma_{\tilde{\mathbf{w}}}(t)$ by ds^2 and dt^2, respectively. Since the metrics and the orientations coincide, there exist a constant s_0 so that $t = s - s_0$, i.e. $\gamma_{\tilde{\mathbf{v}}}(s) = \gamma_{\tilde{\mathbf{w}}}(s - s_0)$. Therefore, for all s,

$$[cos\ s\ v_0 + sin\ s\ v_1] = [cos(s - s_0)\ v_0 + sin(s - s_0)\ v_1],$$

which can hold if there exist a positive real number r such that, for all s,

(2.7)
$$(cos\ s\ v_0 + sin\ s\ v_1) = r(cos(s - s_0)\ v_0 + sin(s - s_0)\ v_1).$$

We will prove that this is equivalent with (2.6).

Indeed, from the above formula we can find easily (for example by putting $s = 0$ and $s = \frac{\pi}{2}$)

$$v_0 = r(cos\ s_0\ w_0 - sin\ s_0\ w_1),$$
$$v_1 = r(cos\ s_0\ w_1 + sin\ s_0\ w_0). \tag{2.8}$$

And using the Euler formula

$$e^{is_0} = cos\ s_0 + isin\ s_0, \tag{2.9}$$

we can prove (2.6) by straightforward calculation.

Conversely, if (2.6) is true, then using (2.9) and identifying the real parts and imaginary parts, we find (2.8), and from here (2.7) by straightforward calculation. □

§3. A holomorphic quadric

Let us consider the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$, and let us denote an equivalence class by $[[z]] \in \mathbb{CP}^n$.

From the previous theorem we obtain:

Corollary. *The relation (2.6) shows that the points $[[v_0 + iv_1]]$ and $[[w_0 + iw_1]]$ in $\mathbb{CP}^n = \mathbb{P}(R^{n+1} \otimes \mathbb{C})$ are equal.*

Now let $\mathbb{RP}^n \subset \mathbb{CP}^n$ denote the set of real points.

Lemma 3.1.
Any point $z \in \mathbb{CP}^n \setminus \mathbb{RP}^n$, it can be represented in the form $z = [[v_0 + iv_1]]$ for some linerly independent (real) vectors v_0, v_1 in R^{n+1}.

Indeed, $z = [[v_0 + iv_1]]$ means that there exists a complex number $\lambda = \lambda_1 + i\lambda_2 \in C\setminus\{0\}$, so that $[[v_0 + iv_1]] = \{\lambda(v_0 + iv_1) = (\lambda_1 v_0 - \lambda_2 v_1) + i(\lambda_2 v_0 - \lambda_1 v_1) | \lambda \in C\setminus\{0\}\}$. If $z \in \mathbb{RP}^n$, then $(\lambda_2 v_0 - \lambda_1 v_1) = 0$, therefore the vectors v_0, v_1 are linearly dependent. If $z \in \mathbb{CP}^n \setminus \mathbb{RP}^n$, then the vectors v_0, v_1 are linearly independent.

Remarks: (i) The plane P_z spanned by the pair $\mathbf{v} = (v_0, v_1)$ and the orientation for which this is an oriented basis are independent of the specific choice of v_0 and v_1 satisfying $z = [[v_0 + iv_1]]$.

(ii) Likewise, the metric on $[P_z]$ for which $\gamma_{\mathbf{v}}$ is a unit speed parametrization does not depend on this choice, but only on z.

(iii) The distiguished basis given by Funk-Busemann's theorem are all equivalent as points in $\mathbb{CP}^n \setminus \mathbb{RP}^n$.

The open manifold $\mathbb{CP}^n \setminus \mathbb{RP}^n$ can be regarded as a space of oriented Riemannian metrics on lines in $\mathbb{S}$.

Indeed, there is a 1-to-1 correspondence between metrics on $[P_z]$ and $z \in \mathbb{CP}^n \setminus \mathbb{RP}^n$. However, this is not the space of all oriented Riemannian metrics on lines in $\mathbb{S}$.

The map $\pi : \mathbb{CP}^n \setminus \mathbb{RP}^n \to \mathbb{G}$ defined by $\pi(z) = [v_0 \wedge v_1]$ is a smooth submersion whose fiber over $[P] \in \mathbb{G}$ consists of a two parameter family of Riemannian metrics on the oriented line $[P]$, where $\mathbf{v} = (v_0, v_1)$ is an oriented basis of P.

The sections of this fibration are

$$s : \mathbb{G} \to \mathbb{CP}^n \setminus \mathbb{RP}^n, \quad s([v_0 \wedge v_1]) = [[\alpha v_0 + \beta v_1]], \tag{3.1}$$

where $\alpha, \beta \in C$.

Recall that Funk's theorem states that among the basis of P there exists some basis, say $\widetilde{\mathbf{v}} = (v, w) \in \mathcal{V}$ for which the parametrization $\gamma_{\widetilde{\mathbf{v}}}$ is unit speed with respect to a fixed Finsler metric with linear geodesics and satisfying $K = 1$. All those basis are in the same equivalence class $\widetilde{z} = [[v + iw]] \in \mathbb{CP}^n \setminus \mathbb{RP}^n$.

We consider now the bundle $\pi : \mathbb{CP}^n \setminus \mathbb{RP}^n \to \mathbb{G}$, where $\pi(z) = [\alpha_{\mathbf{v}}]$. The fibre $\pi^{-1}([P])$ consists on the set of all metrics $z \in \pi^{-1}([P])$ on $[P]$ for which $\gamma_{\mathbf{v}}$ is unit speed parametrization (w.r.t the metric coming from the Finsler structure). Obviously $\widetilde{z} \in \pi^{-1}([P])$.

The section σ that assignes to $[P]$ its Finslerian metric $\widetilde{z} \in \pi^{-1}([P])$ is called *the canonical section* associated to $\Sigma_{\mathbb{S}}$. The section is unique because the point $\widetilde{z} \in \mathbb{CP}^n \setminus \mathbb{RP}^n$ is unique.

Hence,

Theorem 3.2. *[Br2] (i) Let $\Sigma_{\mathbb{S}}$ be a Finsler structure on $\mathbb{S}$ with linear geodesics and whose curvature satisfies K=1. Then the image of the canonical section $\sigma : \mathbb{G} \to \mathbb{CP}^n \setminus \mathbb{RP}^n$ is a (holomorphic) quadric in $\mathbb{CP}^n$.*

(ii) Conversely, if $C \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$ is a smooth quadric with the property that the map $\pi : C \to \mathbb{G}$ is a diffeomorphism onto its image, then there exists a generalized Finsler structure $\Sigma_C \subset T\mathbb{S}$ with the property that $\pi(z)$ endowed with the metric z is a Σ_C-curve.

Furthermore, this generalized Finsler structure has the lines in $\mathbb{S}$ as geodesics and satisfies K=1.

This quadric is equivalent with the complex form of the structure equations for the Riemannian manifold ($\mathbb{G}$). With the standard Euclidean inner product the algebraic way seems to be more suitable for study.

Moreover,

Theorem 3.3. *[Br2]*

The quadric $\sigma(\mathbb{G}) = C \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$, from the Theorem 9.6., can be uniquely written in the form:

$$C = \{[[v]] \in \mathbb{CP}^n \setminus \mathbb{RP}^n | Q(v) = 0\} \tag{3.2}$$

where Q is defined by

$$Q(v) = z_0^2 + e^{ip_1} z_1^2 + ... + e^{ip_n} z_n^2,$$

where $v = (z_0, z_1, ..., z_n)$, and p_i are real numbers satisfying $0 \leq p_1 \leq ... \leq p_n < \pi$.

Any other quadric without real points will be equivalent with this one.

§4. Recovering the Finsler Structure on $\mathbb{S}$.

In the second part of Theorem 3.2 is stipulated the fact that one can recover the Finsler structure on $\mathbb{S}$ starting with the geometrical structure of $\mathbb{G}$. We will see here how is possible to do it concretely in the n-dimensional case.

Recall that if $C \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$ is a complex curve with the property that the map $\tau : C \to \mathbb{S}^*$ is a diffeomorphism onto its image, then there exists a generalized Finsler structure $\Sigma_C \subset T\mathbb{S}$ with the property that $\tau(z)$ endowed with the metric z is a Σ_C-curve.

Furthermore, this generalized Finsler structure has the lines in $\mathbb{S}$ as geodesics and satisfies K=1.

Remark.

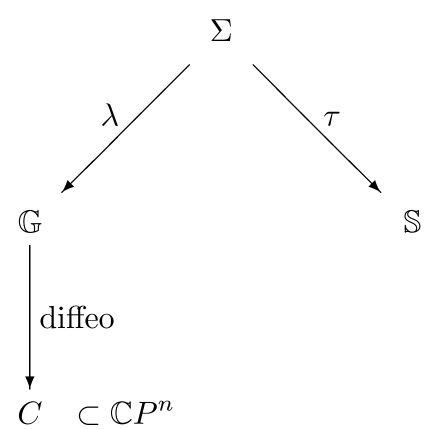
Recall that in A we defined the tangent vectors as the pairs $\gamma'_{\mathbf{v}} = [v, w]$ obtained from the curve $\gamma_{\mathbf{v}}(t) = [v + tw]$. On the other hand, the linear godesics of the classical path geometry on $\mathbb{S}$ were curves $\gamma_{\mathbf{v}}$ in the form $\gamma_{\mathbf{v}}(t) = [v \cdot \cos t + w \cdot \sin t]$. Putting the condition for the two forms of $\gamma_{\mathbf{v}}$ to be the same, i.e. $[v + tw] = [v \cdot \cos t + w \cdot \sin t]$, we obtain $[v + tw] = [\cos t(v + w \cdot \tan t)]$ and because this are classes of $\mathbb{S}$, we need $\cos t > 0$, i.e. a half of circle.

An other immediate consequence is that the Finslerian length of a geodesic is 2π.

We are going to look for a Finsler structure on Σ with linear geodesics, whose manifold of geodesics is $\mathbb{G}$, and $K = 1$.

If we assume the existence of a Finsler strucure like this (theorem 3.2), Funk-Busemann's theorem tell us that there is a special basis, say $\tilde{\mathbf{v}} = (v_0, bv_1 + av_0)$, where a, b are real numbers, $b > 0$, such that the Finslerian length of $\gamma'_{\tilde{\mathbf{v}}}(s)$ is equal to 1.

(4.1)



Start now with a conic $C = \{|[v]| \quad |Q(v) = 0\} \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$, where Q is the quadratic form $Q(v) = v \bullet v = z_0^2 + e^{ip_1} z_1^2 + ... + e^{ip_n} z_n^2$, for $v = (z_0, z_1, ..., z_n) \in V$. We know that all conics in $\mathbb{CP}^n \setminus \mathbb{RP}^n$ are equivalent to this one (Theorem 3.3).

For any basis $\mathbf{v} = (v, w)$, we can consider the curve $\gamma_{\mathbf{v}} : (-a, a) \to \mathbb{S}$ given by $\gamma_{\mathbf{v}}(t) = [v + tw]$, or equivalently, $\gamma_{\mathbf{v}}(t) = [v \cdot \cos t + w \cdot \sin t]$. Therefore, we have to consider the tangent vectors $[v, w] \in T_{[v]}\mathbb{S}$, and the indicatrix of a Finsler structure will include this kind of vectors of unit length. Using this basis we can construct the lines $P_{\mathbf{v}} = [v \wedge w] \in \mathbb{G}$.

Moreover, we have the bundle $\pi : \mathbb{CP}^n \setminus \mathbb{RP}^n \to \mathbb{G}$, by $\pi(z) = [v \wedge w]$, for all $z = |[v + iw]|$.

If we consider the complexified line $|[\alpha\ v + \beta\ w]|$ spaned by v, w, where $\alpha, \beta \in \mathbb{C}^*$, it intersects the conic C in two points $p_i = |[\alpha_i\ v + \beta_i\ w]|$, $i = 1, 2$.

Proposition 4.1. *The point $p = |[\alpha\ v + \beta\ w]| \in \mathbb{CP}^n \setminus \mathbb{RP}^n$ can be always written in the form $p = |[(1 + ia)v + ibw]|$, for some real numbers a, b, with $b \neq 0$.*

Proof. Since p is not real, then α, β cannot vanishes, and $\frac{\alpha}{\beta}$ cannot be real. Therefore, if we consider $\alpha = a_1 + ia_2$, and $\beta = b_1 + ib_2$, then we write

$$p = ||[\alpha\ v + \beta\ w]|| = ||[\beta(\frac{\alpha}{\beta}v + w)]|| = ||[\frac{\alpha}{\beta}v + w||].$$

On the other side

$$\frac{\alpha}{\beta} = \frac{a_1 + ia_2}{b_1 + ib_2} = \frac{a_1b_1 - a_2b_2}{b_1^2 + b_2^2} + i\frac{a_1b_2 + a_2b_1}{b_1^2 + b_2^2},$$

and since $\frac{\alpha}{\beta}$ is not real we have the condition

$$a_1b_2 + a_2b_1 \neq 0.$$

Let us put now $a := \frac{a_1b_1 - a_2b_2}{a_1b_2 + a_2b_1}$, and $b := -\frac{b_1^2 + b_2^2}{a_1b_2 + a_2b_1}$ (is clear that $b \neq 0$). Then we have

$$\frac{\alpha}{\beta} = \frac{1 + ia}{ib},$$

and substituting this in the form of p we have

$$p = ||[\frac{\alpha}{\beta}v + w||] = ||[\frac{1 + ia}{ib}v + w]|| = ||[(1 + ia)v + ibw]||.$$

□

Therefore we have two points of intersection:

$$\begin{aligned} p_1 &= ||[(1 + ia_1)v + ib_1w]|| = ||[v + i(b_1w + a_1v)]||, \\ p_2 &= ||[(1 + ia_2)v + ib_2w]|| = ||[v + i(b_2w + a_2v)]||. \end{aligned} \tag{4.2}$$

And because those points represent different points in $\mathbb{S}$ we can put $b_1 < 0 < b_2$.

Since we constructed them (in the previous Proposition 9.8) from v, w, the coefficients b_1, b_2 can be regarded as analytic functions of v, w, i.e. $b_1 = b_1(v, w), b_2 = b_2(v, w)$.

Now, since $p_i \in \mathbb{CP}^n$, they are the equivalence classes generates from the basis $(v, b_1w + a_1v)$ and $(v, b_2w + a_2v)$, respectively. The corresponding oriented lines are $\pi(p_i) = [v \wedge (b_iw + a_iv)] = [b_iv \wedge w]$.

On the other side, the corresponding tangent vectors can be written in the form $[v, (b_2w + a_2v)] = b_2[v, w] \in T_{[v]}\mathbb{S}$.

From here we can construct the indicatrix from unit vectors with respect to a Finsler structure to be determined. This is equivalent to the condition $b_2(v, w)\ F_C(v, w) = 1$, so we can define the Finsler structure by the formula $F_C(v, w) := \frac{1}{b_2(v, w)}$. Therefore we have to calculate the expression of $b_2(v, w)$ from the equation $Q(p_i) = 0$.

Denoting $Q(v \wedge w) := (v \bullet v)(w \bullet w) - (w \bullet v)^2$, by some algebraic considerations one can see that $Q(v \wedge w) > 0$. Now we start calculating $b_2(v, w)$.

Starting with $Q(p_2) = p_2 \bullet p_2 = 0$, we can find $b_2(v, w)$ by some highschool level calculations in the following way. Expanding the relation $Q(p_2) = 0$, and dividing by $v \bullet v$ we obtain

$$0 = \Big(1 + ia_2 + ib_2 \frac{(v \bullet w)}{(v \bullet v)}\Big)^2 - b_2^2 \frac{Q(v \wedge w)}{(v \bullet v)^2}.$$

After multiplication with -1 and using the formula for difference of squares, we obtain

$$a_2 + b_2 \left(\frac{(v \bullet w)}{(v \bullet v)} + i\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} \right) = i, \tag{4.3}$$

where $\varepsilon = \pm 1$ (means 1 or -1).

But $\left(\frac{(v \bullet w)}{(v \bullet v)} + i\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} \right)$ is a complex number, therefore it can be written in the form $X + iY$, for some real numbers X, Y. Now, relation (4.3) reads $a_2 + b_2(X + iY) = i$, and from here $b_2 Y = 1$. Since $b_2 > 0$ it follows that $Y > 0$. In fact we have already the desired Finsler function in hand. It is $F = Y = Im(X + iY)$. We have to study a litle the implications of ε, and then to finish the computation for $Im(X + iY)$. For the moment we note that the imaginary part of $\left(\frac{(v \bullet w)}{(v \bullet v)} + i\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} \right)$ has to be positive.

The same argument applied to p_1 leads to

$$a_1 + b_1 \left(\frac{(v \bullet w)}{(v \bullet v)} - i\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} \right) = i, \tag{4.4}$$

and the imaginary part of $\left(\frac{(v \bullet w)}{(v \bullet v)} - i\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} \right)$ has to be negative in this case.

Now multiplying (4.3) by b_1, (4.4) by b_2, and by substraction we eliminate the term $\frac{(v \bullet w)}{(v \bullet v)}$. Multiplying the result by i we obtain

$$-(b_1 a_2 - b_2 a_1) i + 2 b_1 b_2 \varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} = (b_1 - b_2).$$

Comparing the real parts of both members of this equality, and taking into account the signs of b_1, b_2 it results that

$$Re\{\varepsilon \frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)}\}$$

should be positive for any basis (v, w). Evaluating this expression at $(v, w) = ((1, 0, 0), (0, 1, 0))$ we obtain $\varepsilon = 1$.

Therefore, (4.3) and (4.4) lead to

$$a_2 + b_2\left(\frac{(v \bullet w)}{(v \bullet v)} + i\frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)}\right) = a_2 + b_2\left(\frac{(v \bullet w)}{(v \bullet v)} - i\frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)}\right) = i.$$

We point out that we discovered already that $F = Y = Im(X + iY)$, where we know now that $\varepsilon = 1$. Therefore, by multiplication with i (if we do not make this multiplication the result will be given using the imaginary part), we find the final formula:

$$F_C(v,w) = Re\{\frac{\sqrt{Q(v \wedge w)}}{(v \bullet v)} - i\frac{(v \bullet w)}{(v \bullet v)}\}. \tag{4.5}$$

We obtain in this way the following result:

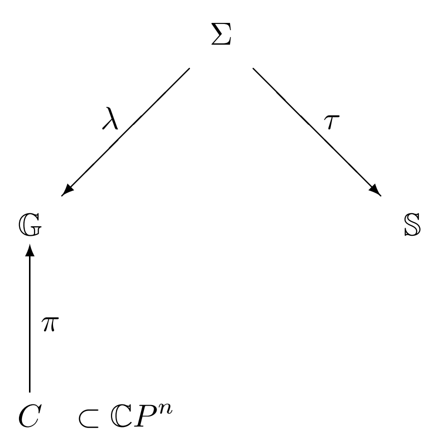
Theorem 4.2. *Let $C \subset \mathbb{C}P^n$ be a conic without real points and let Q be a normalized quadratic form on $V \otimes C$ so that $C = \{[[v]] \in \mathbb{C}P^n \mid Q(v) = 0\}$. Let the inner product of two vectors v and w with respect to Q be denoted by $v \bullet w$.*

Then F_C from (4.5) defines the Finsler metric of the Finsler structure on $\mathbb{S}$ with linear geodesics and K=1 whose canonical section $\sigma : \mathbb{G} \to \mathbb{C}P^n$ has its image equal to C.

Conclusion.

Starting from the fact that $\mathbb{G}$ is the manifold of geodesics of the projectively flat Finsler manifold $(\mathbb{S}, F_C)$, then the above technique can be summarized as follows. Let us consider the diagram

(4.6)



with the same notations as previous.

If we consider a point in $\mathbb{C}P^n$, by applying consecutively the maps π and λ^{-1}, we can obtain a vector in Σ. Namely,

$$[[z]] = [[v + iw]] \in \mathbb{C}P^n \mapsto [v \wedge w] \in \mathbb{G} \mapsto [v,w] \in \Sigma. \tag{4.7}$$

Therefore, if we start with an arbitrary vector in $\mathbb{C}P^n$, the corresponding vector in the indicatrix is $[v,w]$.

Moreover, we know that $\mathbb{G}$ is diffeomorphic to the quadric C. From the algebraic definition of C, a point of the quadric is written in the form $[[v + i(b_2w + a_2v)]]$, and taking into account of (4.7) we have:

$$[[v + i(b_2w + a_2v)]] \in \mathbb{C}P^n \mapsto [v \wedge (b_2w + a_2v)] \in \mathbb{G} \mapsto \\ \mapsto [v, b_2w + a_2v] = b_2[v, w] \in \Sigma \tag{4.8}$$

Hence, the tangent vector to Σ, that corresponds to the point $[[v+i(b_2w+a_2v)]]$ of C, has the form $b_2[v, w] \in \Sigma$.

Putting now the condition $F(b_2[v, w]) = 1$ we obtain

$$F([v, w]) = \frac{1}{b_2}. \tag{4.9}$$

On the other side, from the definition of the quadric C, i.e. the relation $Q(v + i(b_2w + a_2v)) = 0$ one can find explicitely b_2 as a function of v and w.

From (4.9) we can find by the calculations above the explicit form of the fundamental function F.

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