Some remarks on Jacobi stability

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The notion of Jacobi stability for geodesics of a Riemannian or Finslerian manifold can be extended to arbitrary dynamical systems. This is the differential geometric theory of the variational equations for deviation of whole trajectories to nearby ones.

We apply this theory to the Brusselator and Van der Pohl equations, and examine the relationship between the linear stability of steady-states and the stability of transient states.

We interpret the Jacobi stability as the *robustness* of the dynamical system.

1. KCC-theory and Jacobi stability

Let us recall first some basics (see [2], [1]).

We mention that the term "KCC-theory" was coined for the first time by P.L. Antonelli in [1].

Let $(x^1, \ldots, x^n) = (x)$,

$$\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}\right) = \left(\frac{dx}{dt}\right) = \dot{x} \tag{1}$$

and t be $2n+1$ coordinates of an open connected subset Ω of the Euclidean $(2n+1)$-dimensional space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1$. And let us consider a second order differential equation (SODE) of the form

$$\frac{d^2x^i}{dt^2} + g^i(x, \dot{x}, t) = 0, \ i \in \{1, 2, \ldots, n\}, \tag{2}$$

where each function $g^i(x, \dot{x}, t)$ is C^∞ in a neighborhood of some initial conditions $((x)_0, (\dot{x})_0, t_0)$ in Ω.

In order to find the basic differential invariants of the system (2) under the non-singular coordinate transformations

$$\begin{aligned} \bar{x}^i &= f^i(x^1, \ldots, x^n), \ i \in \{1, 2, \ldots, n\}, \\ \bar{t} &= t, \end{aligned} \tag{3}$$

we define the KCC-covariant differential of a contravariant vector field $\xi^i(x)$ on the open subset Ω by

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + \frac{1}{2} g^i_{;r} \xi^r, \tag{4}$$

where the semicolon ";" indicates partial differentiation with respect to $\dot{x}$. The idea of this approach belongs to Kosambi [6], E. Cartan [4] (who corrected Kosambi's work), and S.S. Chern (for the most general version) [5]. The Einstein summation convention will be used throughout.

Using (4), the system (2) becomes

$$\frac{D\dot{x}^i}{dt} = \frac{1}{2}\, g^i_{;r}\dot{x}^r - g^i = \varepsilon^i, \tag{5}$$

where ε^i defined here is a contravariant vector field on Ω and is called the *first* KCC-*invariant*. It is interpreted as an external force [2].

The functions $g^i = g^i(x, \dot{x}, t)$ are 2-homogeneous in $\dot{x}$ if and only if $\varepsilon^i = 0$. In other words, $\varepsilon^i = 0$ is a necessary and sufficient condition for a semispray to be a spray. It is obvious that for the geodesic spray of a Riemannian or Finsler manifold, the first invariant vanishes.

Let us vary the trajectories $x^i(t)$ of (2) into nearby ones according to

$$\bar{x}^i(t) = x^i(t) + \eta\, \xi^i(t), \tag{6}$$

where η denotes a parameter with $|\eta|$ small and where $\xi^i(t)$ are the components of some contravariant vector field defined along the path $x(t)$. Since $\bar{x}$ and x are solutions of (2), it is not difficult to see that $\eta\ddot{\xi} + (\bar{g} - g) = 0$, where $\bar{g} - g := g(t, x + \eta\xi, \dot{x} + \eta\dot{\xi}) - g(t, x, \dot{x})$. View $\bar{g} - g$ as a function of η and apply the mean value theorem. This enables us to cancel off the η which multiplies $\ddot{\xi}$. Finally, take the limit $\eta \to 0$ to obtain the variational equations

$$\frac{d^2\xi^i}{dt^2} + g^i{}_{;r}\,\frac{d\xi^r}{dt} + g^i{}_{,r}\xi^r = 0, \tag{7}$$

where the comma "," indicates partial differentiation with respect to x^r.

Using now the KCC-covariant differential (4), one obtains (7) in the covariant form

$$\frac{D^2\xi^i}{dt^2} = P^i_r\xi^r, \tag{8}$$

where

$$P^i_j = -g^i{}_{,j} - \frac{1}{2}g^r g^i{}_{;r;j} + \frac{1}{2}\dot{x}^r g^i{}_{,r;j} + \frac{1}{4}g^i{}_{;r}g^r{}_{;j} + \frac{1}{2}\frac{\partial g^i{}_{;j}}{\partial t} \tag{9}$$

is called the *second* KCC-*invariant* of the system (2), or *deviation curvature tensor*.

Note that (8) is the *Jacobi field equation* whenever system (2) are the geodesic equations in either Finsler or Riemannian geometry. The notion of Jacobi stability for SODE's is thus a generalization of that for the geodesics of a Riemannian or Finsler manifold. This justifies the usage of the term *Jacobi stability* for KCC-Theory.

We are interested in the "*focusing tendency*" of trajectories of (2) in a vecinity of a point $x^i(t_0)$ of it. For simplicity we will consider $t_0 = 0$.

Let us consider the trajectories $x^i = x^i(t)$ of (2) as curves in the Euclidean space $(I\!R^n, < \cdot, \cdot >)$, where $< \cdot, \cdot >$ is the canonical inner product of $I\!R^n$, and let us impose to the deviation vector ξ the initial conditions

$$\xi(0) = O, \qquad \dot{\xi}(0) = W \neq O,$$

where $O \in I\!R^n$ is the null vector.

We consider now an "adapted" inner product $<< \cdot, \cdot >>$ to the deviation tensor ξ by

$$<< X, Y >> := \frac{1}{< W, W >} \cdot < X, Y >$$

for any vectors $X, Y \in I\!R^n$. Obviously, $||W||^2 := << W, W >> = 1$.

Then, we can imagine the "*focusing tendency*" of trajectories around 0 as:

- trajectories are **bunching together** if $||\xi(t)||^2 < t^2$, $t \approx 0^+$,
- trajectories are **dispersing** if $||\xi(t)||^2 > t^2$, $t \approx 0^+$

(see [3] for a detailed explanation of the Finslerian geodesics' behavior).

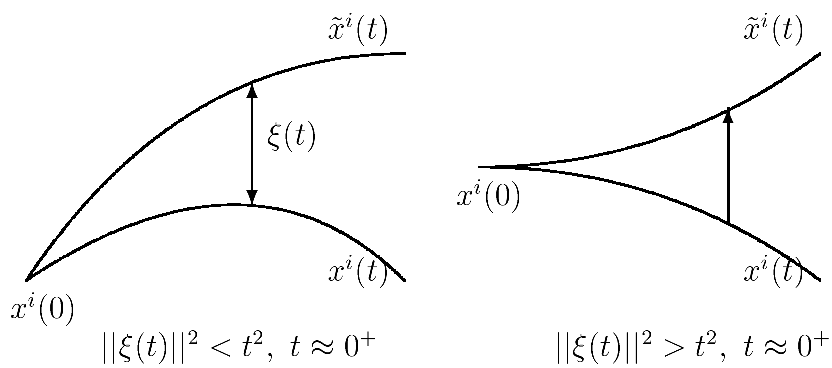


Figure 1. Behavior of trajectories near zero.

It follows:

Lemma 1.1

Let us consider the SODE (2) *and its variation* (6) *with the tensor* $\xi(t)$ *satisfying the initial conditions:*

$$\xi(0) = O, \qquad \dot{\xi}(0) = W \neq O,$$

where $O \in I\!R^n$ *is the null vector.*

Then, for $t \approx 0^+$, *the trajectories of* (2) *are*

- *bunching together if and only if the real part of the eigenvalues of* $P^i_j(0)$ *are strict negative,*
- *dispersing if and only if the real part of the eigenvalues of* $P^i_j(0)$ *are strict positive.*

Now the notion of Jacobi stability can be defined as follows.
Definition 1.1.
If the SODE (2) satisfies the initial conditions:

$$||x^i(t_0) - \tilde{x}^i(t_0)|| = 0, \; ||\dot{x}^i(t_0) - \dot{\tilde{x}}^i(t_0)|| \neq 0,$$

with respect to the norm $||\cdot||$ induced by a positive definite inner product, then **the trajectories of** (2) **are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P_i^j are strict negative everywhere, and Jacobi unstable, otherwise.**

Remark. The third, fourth and fifth KCC-invariants are:

$$R^i_{jk} = \frac{1}{3}(P^i_{j;k} - P^i_{k;j}), \quad B_j{}^i{}_{k\ell} = R^i_{jk;\ell}, \quad D_j{}^i{}_{k\ell} = g^i_{;j;k;\ell}. \tag{10}$$

A basic result of the KCC-theory is the following

Proposition 1.2 ([2]) *Two* SODE's *of form* (2) *on* Ω *can be locally transformed, relative to* (3)*, one into other, if and only if their five* KCC-*invariants* ε^i, P^i_j, R^i_{jk}, $B_j{}^i{}_{k\ell}$, $D_j{}^i{}_{k\ell}$ *are equivalent tensors. In particular, there are local coordinates* $(\bar{x})$ *for which* $g^i(\bar{x}, \dot{\bar{x}}, t) = 0$ *if and only if all five* KCC-*tensors vanish.*

Remark. It would be interesting to correlate Linear stability with Jacobi stability. In other words, to compare the signs of the eigenvalues of the Jacobian matrix J at a fixed point with the signs of the eigenvalues of the deviation curvature tensor P_i^j evaluated at the same point. Even though this should be possible in the general case, we give here only a discussion for the 2-dimensional case.

Let us consider the following system of ODE:

$$\frac{du}{dt} = f(u, v) \qquad \frac{dv}{dt} = g(u, v) \tag{11}$$

such that the point $(0, 0)$ is a fixed point, i.e. $f(0, 0) = g(0, 0) = 0$. In general, even though the fixed point is (u_0, v_0), by the change of variables $\bar{u} = u - u_0$, $\bar{v} = v - v_0$, one can always obtain $(0, 0)$ as a fixed point. We denote by J the Jacobian matrix of (11), i.e.

$$J(u, v) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \tag{12}$$

where the subscripts indicate partial derivatives with respect to u and v.

The oscillation conditions of (11) are $trA > 0$ and $detA > 0$, where A is the Jacobian J evaluated at the fixed point $(0, 0)$, i.e. $J|_{(0,0)} = A$. Using the characteristic equation and some elementary algebra, we see that only the following *periodic behavior of* (11) *are possible*:

1. The point $(0, 0)$ is an unstable spiral, i.e.

$$\Delta < 0, \qquad S > 0.$$

2. The point $(0,0)$ is an unstable node, i.e.

$$\Delta > 0, \qquad S > 0, \qquad P > 0,$$

where $\Delta = (f_u - g_v)^2 + 4 f_v g_u = (trA)^2 - 4\, detA$, $S = f_u + g_v = trA$, and $P = f_u g_v - f_v g_u = detA$ are the discriminant of the characteristic equation, the sum and product of the eigenvalues, respectively.

By elimination of one of the variables, we can transform (11) into a SODE of the form (12) and compute the deviation curvature. It does not matter which of the two variables we eliminate, because the resulting differential equations are topologically conjugate ([8]).

Let us relabel v as x, and $g(u,v)$ as y, and let us assume that $g_u|_{(0,0)} \neq 0$. The variable to be eliminated, u in this case, should be chosen such that this condition holds. Since $(u,v) = (0,0)$ is a fixed point, the Theorem of Implicit functions implies that the equation $g(u,x) - y = 0$ has a solution $u = u(x,y)$ in the vicinity of $(x,y) = (0,0)$. Now, $\ddot{x} = \dot{g} = g_u\, f + g_v\, y$. Hence we obtain an autonomous one-dimensional case of the SODE (2), namely $\ddot{x}^1 + g^1 = 0$, where

$$g^1(x,y) = -g_u(u(x,y),x)\, f(u(x,y),x) - g_v(u(x,y),x)\, y. \tag{13}$$

Evaluating now the curvature tensor P^1_1 from (9) at the fixed point $(0,0)$, we obtain after some computation:

$$4\, P^1_1|_{(0,0)} = -4 g^1_{,1}|_{(0,0)} + (g^1_{;1})^2|_{(0,0)} = \Delta := (trA)^2 - 4\, detA. \tag{14}$$

It can be checked that the second equality would not change if one had used the first equation of (11) to eliminate the variable v instead. Now, straightforward calculations give the following.

Theorem 1.3 *Let us consider the ODE* (11) *with the fixed point p=(0,0) such that* $g_u|_{(0,0)} \neq 0$.

Then, the Jacobian J estimated at p has complex eigenvalues with positive real parts if and only if p is a Jacobi stable point.

Corollary 1.4

(i) If p is an unstable spiral , then it is Jacobi stable. Conversely, if $trA > 0$ *and p is Jacobi stable, then it is an unstable spiral.*

(ii) If p is an unstable node, then it is Jacobi unstable. Conversely, if we have $trA > 0$, $detA > 0$, *and p is Jacobi unstable, then it is an unstable node.*

2. Worked Examples

2.1. The Brusselator

We present a first example of a dynamical system that has a limit cycle behavior, but which is not Jacobi stable in its entirety. In other words, the Brusselator is not robust in its entirety.

The Brusselator is a simple biological system proposed by Prigone and Lefever in 1968. The chemical reactions are:

$$\begin{aligned} A &\xrightarrow{k_1} X, \\ B + X &\xrightarrow{k_2} Y + D, \\ 2X + Y &\xrightarrow{k_3} 3X, \\ X &\xrightarrow{k_4} E \end{aligned} \tag{15}$$

where the k's are rate constants, and the reactant concentrations of A and B are kept constant. Then, the law of mass action leads to the following system of differential equations:

$$\begin{aligned} \frac{du}{dt} &= 1 - (b+1)u + au^2v =: f(u,v) \\ \frac{dv}{dt} &= bu - au^2v =: g(u,v), \end{aligned} \tag{16}$$

where u and v correspond to the concentrations of X and Y, respectively, and a, and b are positive constants.

2.1.1. Linear stability analysis

From the equations (16) it follows that there is a unique fixed point $S(u_0 = 1, v_0 = \frac{b}{a})$.

We are interested in conditions on the parameters a and b for oscillations. They read

$$trA > 0 \quad \text{and} \quad detA > 0.$$

Under these conditions, we recall that

- if $\Delta > 0$, then the point S is said an *unstable node*, and
- if $\Delta < 0$, then the point S is said an *unstable spiral*,

where $\Delta = (trA)^2 - 4\,detA$.

Here, the Jacobian is given by

$$\begin{aligned} A &= J|_{(u_0,v_0)} \\ &= \begin{pmatrix} -(b+1) + 2auv & au^2 \\ b - 2auv & -au^2 \end{pmatrix} |_{u_0=1, v_0=\frac{b}{a}} \\ &= \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}, \end{aligned} \tag{17}$$

hence

$$trA = b - 1 - a, \quad detA = a.$$

Since a and b are positive constants, the condition for oscillation of the Brusselator is

$$b - 1 - a > 0, \tag{18}$$

in which case $\Delta > 0$ (resp. $\Delta < 0$) if and only if $b-1-a > 2\sqrt{a}$ (resp. $b-1-a < 2\sqrt{a}$).

Using these notations, one gets:

Proposition 2.1 *The fixed point $S(1, b/a)$ is*

- *an unstable node if and only if* $b > (\sqrt{a}+1)^2$, *and*
- *an unstable spiral if and only if* $b < (\sqrt{a}+1)^2$.

Moreover, the point $b_c = a+1$ *is a critical point in the sense of Hopf, and for* $b > b_c$ *the system exhibits a limit cycle behavior.*

2.1.2. Jacobi stability analysis

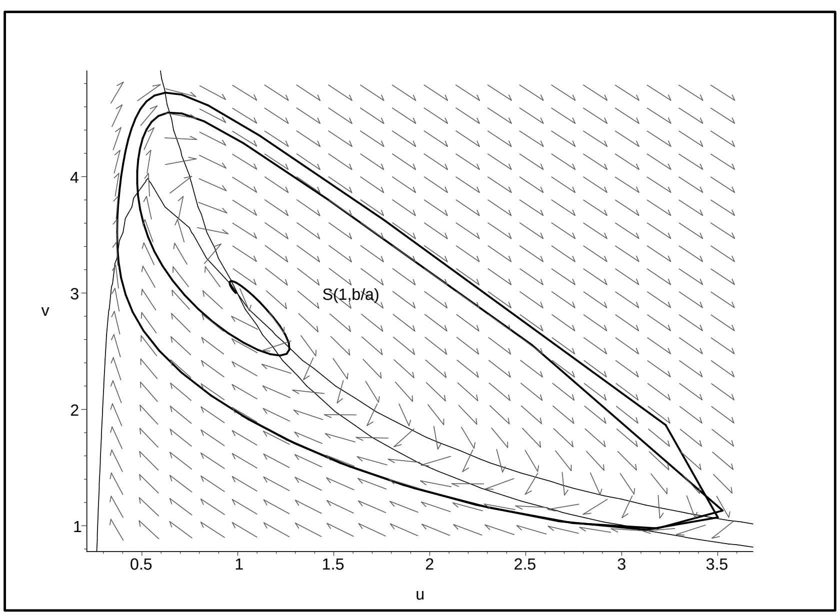
The bifurcation analysis does not give any information about the stability of the periodic solutions. We are going to estimate the stability of such a solution by means of Jacobi stability.

First, let us remark that because of $trA > 0$, and $detA > 0$, it follows from Collorary 1.4 that:

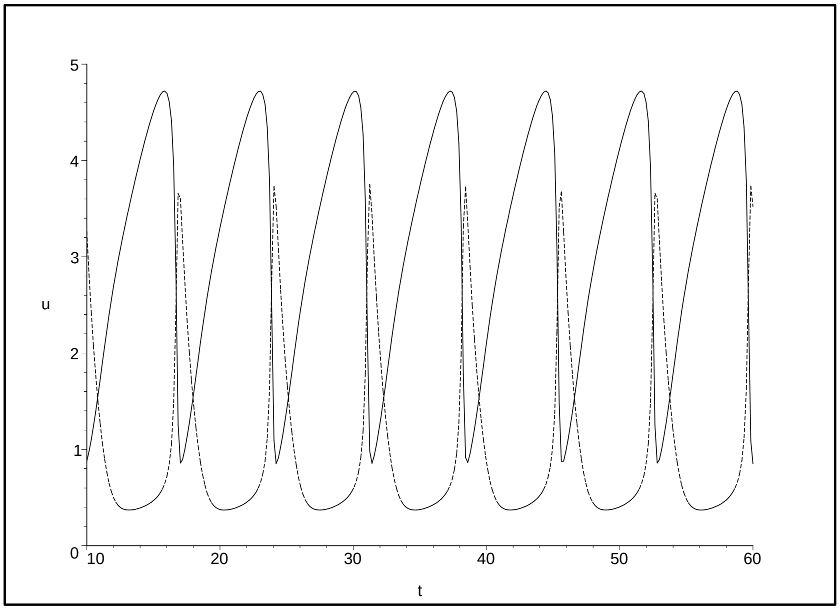
Proposition 2.2

- *If the fixed point* $S(1, \frac{b}{a})$ *is an unstable spiral, then it is in the Jacobi stability region. Conversely, if it is in the Jacobi stability region, and* $b > a+1$, *then it is an unstable spiral.*
- *If the fixed point* $S(1, \frac{b}{a})$ *is an unstable node, then it is in the Jacobi instability region. Conversely, if it is in the Jacobi instability region, and* $b > a+1$, *then it is an unstable node.*

This can also be seen directly from $4P^1_1|_{(1,\frac{b}{a})} = \Delta = (trA)^2 - 4\,detA$.

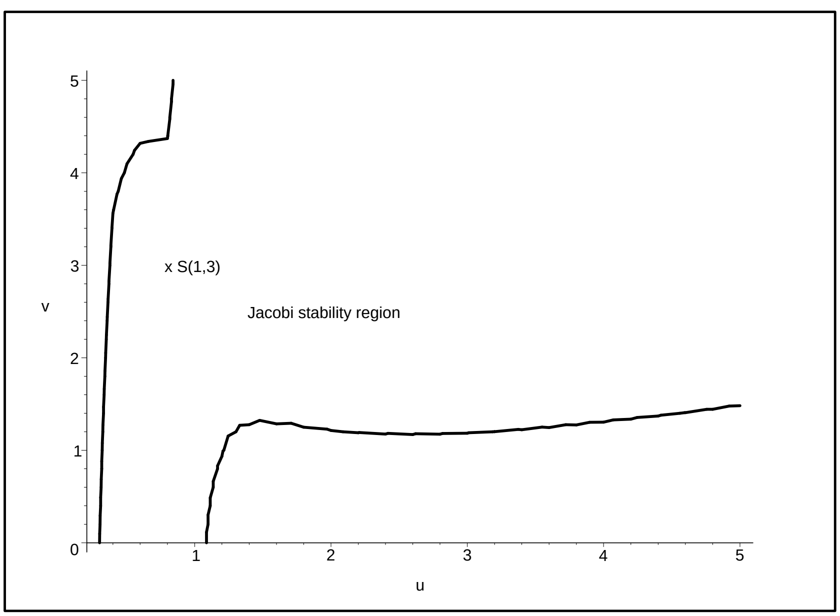


(a) Limit cycle behavior around an unstable spiral fixed point of the solutions of the Brusselator.

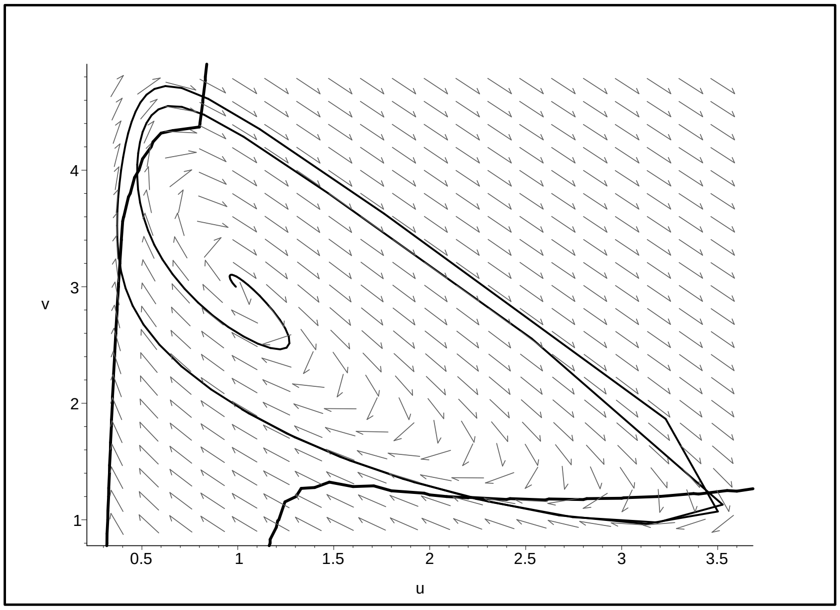


(b) Numerical solution (from integrating the equations for $a = 1$, $b = 3$).

Figure 2. Limit cycle of the Brusselator.



(a) The Jacobi stability region.



(b) The limit cycle cannot lie inside the Jacobi stability region.

Figure 3. Jacobi stability and the Brusselator.

The above discussion shows that for the Brusselator, we have a region of Jacobi stability and a region of Jacobi instability in the plane, these zones not being necessarily connected. We would like to study now the Jacobi stability of a solution as a whole. In other words, we would like to know whether it is possible to construct the confined set *inside* the Jacobi stability region.

It can be seen from Figure 3 that this is impossible for the Brusselator. Even though the solution stays mostly in the Jacobi stability region, there are points in the Jacobi instability region.

Indeed, for instance the point $(\frac{1}{b+1}, 0)$, i.e. the intersection of the nullcline $f = 0$ with the abscissa, belongs to the Jacobi instability region, for any $a, b > 0$. This can be seen by calculating

$$P_1^1|_{(\frac{1}{b+1},0)}$$
$$= b^6 + 6b^5 + 15b^4 + (2a + 20)b^3 + (20 + 14)b^2 + 4b + (a - b - 1)^2 > 0.$$

We can see that there is a Jacobi bifurcation point, b=2, where the steady-state solution changes its Jacobi stability (see Table 1 below.)

value of b	0	2	4
Linear stability	stable	unstable	unstable (limit cycle)
Jacobi stability	stable	stable	unstable

Table 1. Linear stability vs. Jacobi stability analysis of the steady-state of the Brusselator.

It follows that *we cannot construct a confined set in the interior of the Jacobi stability region.* Therefore, in the case of Brusselator, the single fixed point S belongs to the Jacobi stability zone, but its periodic trajectories are not always Jacobi stable.

Still, in an open region around S, the transient trajectories are Jacobi stable, so we can conclude that in this region the system is robust and it becomes fragile outside it. This conclusion is in accord with the usual stability analysis done with Floquet theory ([8]).

2.2. Van der Pohl equations

Van der Pohl equations form a classical example of an oscillatory system. This system also exhibits a limit cycle behavior, but in this case, there are regions in the parameter space where the periodic trajectories are Jacobi stable in their entirety. In other words, for these regions, the *periodic solutions of the Van der Pohl equations are robust.*

We start directly with the system of differential equations, which reads:

$$\begin{aligned} \frac{du}{dt} &= v := f(u,v) \\ \frac{dv}{dt} &= a(1-u^2)v - u := g(u,v) \end{aligned} \tag{19}$$

where $a \in I\!R$.

2.2.1. Linear stability

The unique fixed point is $O(u_0 = 0, v_0 = 0)$, and the Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ -2auv - 1 & a(1-u^2) \end{pmatrix}. \tag{20}$$

It follows that

$$A = J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \tag{21}$$

with the characteristic equation $\lambda^2 - a\lambda + 1 = 0$. We have $trA = a$ and $detA = 1$. The eigenvalues of the system are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}a + \frac{1}{2}\sqrt{a^2-4} \\ \lambda_2 &= \frac{1}{2}a - \frac{1}{2}\sqrt{a^2-4}. \end{aligned} \tag{22}$$

The condition for oscillations, $trA > 0$ and $detA > 0$, are equivalent to $a > 0$. On the other hand, the sum and product of the eigenvalues being $S = a$ and $P = 1$, respectively, from the oscillation condition we obtain $S > 0$, $P > 0$.

Under these conditions, we recall that

- if $\Delta > 0$, then the point S is an *unstable node*, and
- if $\Delta < 0$, then the point S is an *unstable spiral*,

where $\Delta = (trA)^2 - 4\,detA$.
Since $\Delta = a^2 - 4$, it follows that:

- if $a \in (2, \infty)$, then the fixed point $O(0,0)$ is an *unstable node*, and
- if $a \in (0, 2)$, then the fixed point $O(0,0)$ is an *unstable spiral*.

The following result is known ([7]):

Proposition 2.3 *The solution of the system* (19) *is periodic for any* $a \in (0, 2)$. *Moreover, here the system has a limit cycle behavior.*

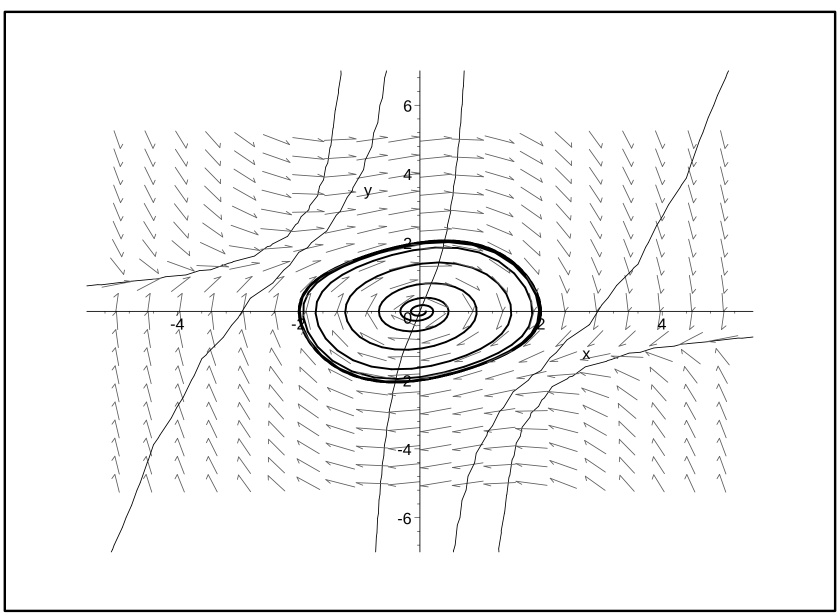
2.2.2. Jacobi stability

From Collorary 1.4 it follows that:

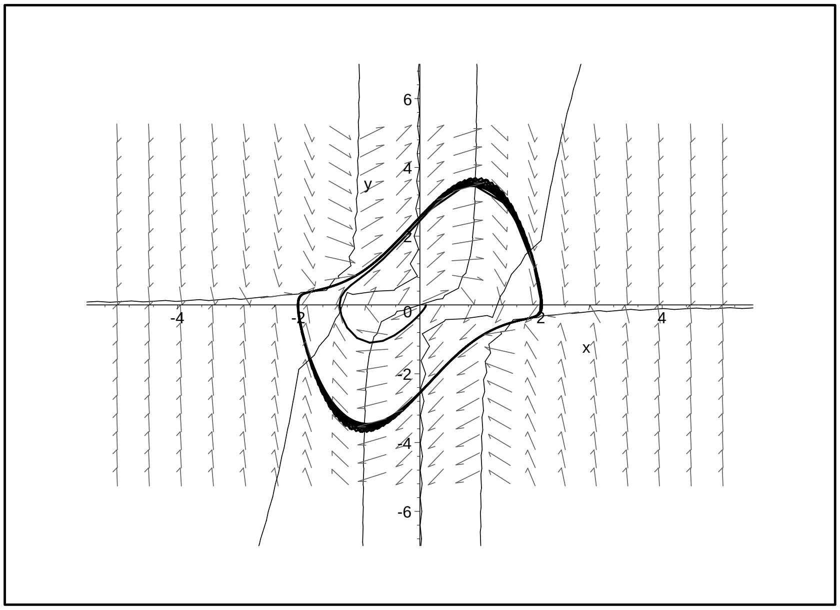
Proposition 2.4 *The fixed point O(0,0) is in the Jacobi stability region if and only if* $a \in (0, 2)$.

This can also be seen by a direct calculation of the deviation curvature:

$$P_1^1 = -axy - 1 + \frac{1}{4}a^2(1 - x^2)^2. \tag{23}$$



(a) For small values of a, the limit cycle lies in the interior of the Jacobi stability region.



(b) For large a, the limit cycle does not lie inside the Jacobi stability region.

Figure 4. The limit cycle of the Van der Pohl equations.

Numerical simulations show that for small values of the parameter $a \in (0, 2)$, for example $a \in (0, 0.4)$, the limit cycle lies in the interior of the Jacobi stability region (see Figure 4 (a)). For this range of parameter, the system exhibits a *robust periodic behavior*. However, for large values, near 2, the limit cycle is no longer contained in the Jacobi stability region (Figure 4 (b)).

In other words, there is a "Jacobi bifurcation" value for a generic solution near 0.4, but we haven't yet for the moment an analytical method to find its value.

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