# RANDERS METRICS ON $\mathbb{CP}^2$

**Abstract**

The Complex projective space $\mathbb{CP}^2$ is a classical example of Einstein metric in Riemannian geometry. Moreover, beside this property, it has other interesting geometrical properties: it is a symmetric space, and a $C_\pi$ manifold. We would like to know if there is an Einstein metric of Raders type on $\mathbb{CP}^2$ with symilar properties. Based on some the generalizatiion of Zermelo navigation problem for Finsler manifolds ([4]) we construct such Randers metric on $\mathbb{CP}^2$ and study some of its geometrical properties.

**MSC 2000: 53C20, 53C60**
**Key Words:** Randers spaces, Einstein equations

## 1 Riemannian Geometry of $\mathbb{CP}^2$

We recall that the complex projective space can be written as

$$\mathbb{CP}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C} = S^5/S^1,$$

where $\mathbb{C}$ act by complex multiplication and $S^5$ and $S^1$ is the 5-dimensional, 1-dimensional unit sphere in $\mathbb{R}^6$, $\mathbb{R}^2$, respectively.

If we write the metric on $S^5$ as

$$ds^2_{S^5} = dr^2 + \sin^2(r)ds^2_{S^3} + \cos^2(r)d\theta,$$

where $ds^2_{S^5}$ and $ds^2_{S^3}$ are the standard Riemannian metrics on the unit spheres $S^5$ and $S^3$ respectively then, we consider that the $S^1$ action on $S^5$ acts on $S^3$ and $S^1$ separately.

It follows that $\mathbb{CP}^2$ can be written as

$$\mathbb{CP}^2 = [0, \frac{\pi}{2}] \times ((S^3 \times S^1)/S^1), \tag{1.1}$$

and its metric is

$$dr^2 + \sin^2(r)(\cos^2(r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2), \tag{1.2}$$

where $\{\sigma^1, \sigma^2, \sigma^3\}$ is the canonical frame on $S^3$. More generally, we can consider a metric on $I \times S^3$ of the form

$$\breve{a} = dr^2 + \varphi^2(r)(\psi^2(r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2). \tag{1.3}$$

We also recall that because of the structure of Lie group of $S^3$, its canonical frame $\{\sigma^1, \sigma^2, \sigma^3\}$ satisfies:

$$d\sigma^1 = 2\sigma^2 \wedge \sigma^3, \quad d\sigma^2 = 2\sigma^3 \wedge \sigma^1, \quad d\sigma^3 = 2\sigma^1 \wedge \sigma^2. \tag{1.4}$$

We will construct in the following a coframe on $\mathbb{CP}^2$ orthonormal with respect to the Riemannian metric $\breve{a}$ given in (1.3). We put

$$\theta^1 := \mu\sigma^1, \quad \theta^2 := \varphi\sigma^2, \quad \theta^3 := \varphi\sigma^3, \quad \theta^4 := dr, \tag{1.5}$$

where $\mu := \varphi\psi$. It is obvious from (1.3) that $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is an orthonormal coframe with respect to $\breve{a}$, i.e.

$$\breve{a} = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + (\theta^4)^2,$$

and let us denote by $\{F_1, F_2, F_3, F_4\}$ the dual orthonormal frame, i.e. $\theta^i(F_j) = \delta^i_j$.

Let us denote by ${\theta_p}^q$ , $p, q = 1, .., 4$, the Riemannian connection 1-forms of the Riemannian metric $\breve{a}$. They satisfy the structure equations

$$\begin{aligned} d\theta^q &= \theta^p \wedge {\theta_p}^q, \\ \theta_{pq} + \theta_{qp} &= 0. \end{aligned} \tag{1.6}$$

Since $h_{ij} = \delta_{ij}$, we have $\theta_{pq} = \delta_{pr}{\theta_q}^r = {\theta_p}^q$.

Using (1.5) in (1.4) we obtain easily

$$
\begin{aligned}
d\theta^1 &= \frac{\mu'}{\mu}\theta^4 \wedge \theta^1 + 2\frac{\mu}{\varphi^2}\theta^2 \wedge \theta^3, \\
d\theta^2 &= \frac{\varphi'}{\varphi}\theta^4 \wedge \theta^2 + \frac{2}{\mu}\theta^3 \wedge \theta^1, \\
d\theta^3 &= \frac{\varphi'}{\varphi}\theta^4 \wedge \theta^3 + \frac{2}{\mu}\theta^1 \wedge \theta^2, \\
d\theta^4 &= 0.
\end{aligned} \tag{1.7}
$$

**Proposition 1.1** *The Levi-Civita connection 1-forms of the Riemannian metric ă are given by*

(1.8)

$$
\begin{pmatrix} \theta_1{}^1 & \theta_1{}^2 & \theta_1{}^3 & \theta_1{}^4 \\ \theta_2{}^1 & \theta_2{}^2 & \theta_2{}^3 & \theta_2{}^4 \\ \theta_3{}^1 & \theta_3{}^2 & \theta_3{}^3 & \theta_3{}^4 \\ \theta_4{}^1 & \theta_4{}^2 & \theta_4{}^3 & \theta_4{}^4 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\mu}{\varphi^2}\theta^3 & \frac{-\mu+2\mu\varphi}{\varphi^2}\theta^2 & -\frac{\mu'}{\mu}\theta^1 \\ \frac{\mu}{\varphi^2}\theta^3 & 0 & -\frac{\mu+2\mu\varphi^2}{\mu\varphi^2}\theta^1 & -\frac{\varphi'}{\varphi}\theta^2 \\ -\frac{-\mu+2\mu\varphi}{\varphi^2}\theta^2 & \frac{\mu+2\mu\varphi^2}{\mu\varphi^2}\theta^1 & 0 & -\frac{\varphi'}{\varphi}\theta^4 \\ \frac{\mu'}{\mu}\theta^1 & \frac{\varphi'}{\varphi}\theta^2 & \frac{\varphi'}{\varphi}\theta^4 & 0 \end{pmatrix}.
$$

<u>*Proof.*</u> Firstly, for simplicity let us put

$$
\begin{pmatrix} \theta_1{}^1 & \theta_1{}^2 & \theta_1{}^3 & \theta_1{}^4 \\ \theta_2{}^1 & \theta_2{}^2 & \theta_2{}^3 & \theta_2{}^4 \\ \theta_3{}^1 & \theta_3{}^2 & \theta_3{}^3 & \theta_3{}^4 \\ \theta_4{}^1 & \theta_4{}^2 & \theta_4{}^3 & \theta_4{}^4 \end{pmatrix} = \begin{pmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{pmatrix}. \tag{1.9}
$$

Since $A, B, C, D, E, F$ are 1-forms, they can be written in the frame $\theta^1, \theta^2, \theta^3, \theta^4$, as follows

$$
\begin{aligned}
A &= A_1\theta^1 + A_2\theta^2 + A_3\theta^3 + A_4\theta^4, \\
B &= B_1\theta^1 + B_2\theta^2 + B_3\theta^3 + B_4\theta^4, \\
C &= C_1\theta^1 + C_2\theta^2 + C_3\theta^3 + C_4\theta^4, \\
D &= D_1\theta^1 + D_2\theta^2 + D_3\theta^3 + D_4\theta^4, \\
E &= E_1\theta^1 + E_2\theta^2 + E_3\theta^3 + E_4\theta^4,
\end{aligned} \tag{1.10}
$$

where $A_i, B_i, C_i, D_i$, $i = 1, ..., 4$ are differentialble functions on $\mathbb{CP}^2$.

With these notations, the structure equations (1.6) become

$$
\begin{aligned}
d\theta^1 &= \qquad\qquad -\theta^2 \wedge A - \theta^3 \wedge B - \theta^4 \wedge C \\
d\theta^2 &= \theta^1 \wedge A \qquad\qquad - \theta^3 \wedge D - \theta^4 \wedge E \\
d\theta^3 &= \theta^1 \wedge B + \theta^2 \wedge D \qquad\qquad - \theta^4 \wedge F \\
d\theta^4 &= \theta^1 \wedge C + \theta^2 \wedge E + \theta^3 \wedge F.
\end{aligned} \tag{1.11}
$$

Using now (1.7), formulas (1.11) become

$$
\begin{aligned}
\frac{\mu'}{\mu}\theta^4 \wedge \theta^1 + 2\frac{\mu}{\varphi^2}\theta^2 \wedge \theta^3 &= \qquad\qquad -\theta^2 \wedge A - \theta^3 \wedge B - \theta^4 \wedge C \\
\frac{\varphi'}{\varphi}\theta^4 \wedge \theta^2 + \frac{2}{\mu}\theta^3 \wedge \theta^1 &= \theta^1 \wedge A \qquad\qquad - \theta^3 \wedge D - \theta^4 \wedge E \\
\frac{\varphi'}{\varphi}\theta^4 \wedge \theta^3 + \frac{2}{\mu}\theta^1 \wedge \theta^2 &= \theta^1 \wedge B + \theta^2 \wedge D \qquad\qquad - \theta^4 \wedge F \\
0 &= \theta^1 \wedge C + \theta^2 \wedge E + \theta^3 \wedge F.
\end{aligned} \tag{1.12}
$$

We will calculate each formula from (1.12) substituting (1.10) in the right hand side.

First formula of (1.12) reads

$$
\begin{aligned}
&\frac{\mu'}{\mu}\theta^4 \wedge \theta^1 + 2\frac{\mu}{\varphi^2}\theta^2 \wedge \theta^3 = -A_1\theta^2 \wedge \theta^1 + (-A_3 + B_2)\theta^2 \wedge \theta^3 + \\
&+(-A_4 + C_2)\theta^2 \wedge \theta^4 - B_1\theta^3 \wedge \theta^1 + (-B_4 + C_3)\theta^3 \wedge \theta^4 - C_1\theta^4 \wedge \theta^1.
\end{aligned} \tag{1.13}
$$

And by identification we obtain

$$
\begin{aligned}
&A_1 = 0, && B_1 = 0, \\
&-A_3 + B_2 = \frac{2\mu}{\varphi^2}, && B_4 = C_3, \\
&-A_4 + C_2 = 0, && C_1 = -\frac{\mu'}{\mu}.
\end{aligned} \tag{1.14}
$$

In the same way, second formula of (1.12) reads

$$
\begin{aligned}
\frac{\varphi'}{\varphi}\theta^4 \wedge \theta^2 + \frac{2}{\mu}\theta^3 \wedge \theta^1 &= A_2\theta^1 \wedge \theta^2 + (A_3 + D_1)\theta^1 \wedge \theta^3 + (A_4 + E_1)\theta^1 \wedge \theta^4 \\
&\quad -D_2\theta^3 \wedge \theta^2 + (-D_4 + E_3)\theta^3 \wedge \theta^4 - E_2\theta^4 \wedge \theta^2,
\end{aligned} \tag{1.15}
$$

and by identification we obtain

$$A_2 = 0, \qquad D_2 = 0,$$
$$A_3 + D_1 = -\frac{2}{\mu}, \qquad D_4 = E_3, \tag{1.16}$$
$$A_4 + E_1 = 0, \qquad E_2 = -\frac{\varphi'}{\varphi}.$$

In the same way, third formula of (1.12) reads

(1.17

$$\frac{\varphi'}{\varphi}\theta^4 \wedge \theta^3 + \frac{2}{\mu}\theta^1 \wedge \theta^2 = (B_2 - D_1)\theta^1 \wedge \theta^2 + B_3\theta^1 \wedge \theta^3 + (B_4 + F_1)\theta^1 \wedge \theta^4$$
$$+D_3\theta^2 \wedge \theta^3 + (D_4 + F_2)\theta^2 \wedge \theta^4 - F_3\theta^4 \wedge \theta^3,$$

and by identification we obtain

$$B_2 - D_1 = \frac{2}{\mu}, \qquad B_3 = 0,$$
$$B_4 + F_1 = 0, \qquad D_3 = 0, \tag{1.18}$$
$$D_4 + F_2 = 0, \qquad F_3 = -\frac{\varphi'}{\varphi}.$$

Finally, the forth formula of (1.12) reads

$$0 = (C_2 - E_1)\theta^1 \wedge \theta^2 + (C_3 - F_1)\theta^1 \wedge \theta^3 + C_4\theta^1 \wedge \theta^4 \tag{1.19}$$
$$+(E_3 - F_2)\theta^2 \wedge \theta^3 + E_4\theta^2 \wedge \theta^4 + F_4\theta^4 \wedge \theta^3,$$

and by identification we obtain

$$C_2 - E_1 = 0, \qquad C_3 - F_1 = 0,$$
$$C_4 = 0, \qquad E_3 - F_2 = 0, \tag{1.20}$$
$$E_4 = 0, \qquad F_4 = 0.$$

From the relations (1.14), (1.16), (1.18), (1.20) all the coefficients can be uniquely determined. For example, we have

$$-A_4 + C_2 = 0$$
$$A_4 + E_1 = 0 \tag{1.21}$$
$$C_2 - E_1 = 0.$$

It follows

$$A_4 = C_2 = E_1 = 0. \tag{1.22}$$

In the same way,

$$\begin{aligned} -A_3 + B_2 &= \frac{2\mu}{\varphi^2} \\ A_3 + D_1 &= -\frac{2}{\mu} \\ B_2 - D_1 &= \frac{2}{\mu} \end{aligned} \tag{1.23}$$

implies

$$A_3 = -B_2 = -\frac{\mu}{\varphi^2}, \quad D_1 = \frac{\mu}{\varphi^2} - \frac{2}{\mu}. \tag{1.24}$$

Finally, from

$$\begin{aligned} B_4 + F_1 = 0 \\ B_4 - C_3 = 0 \\ C_3 - F_1 = 0 \end{aligned} \tag{1.25}$$

we get

$$B_4 = C_3 = F_1 = 0 \tag{1.26}$$

and from

$$\begin{aligned} D_4 + F_2 = 0 \\ E_3 - F_2 = 0 \\ D_4 - E_3 = 0 \end{aligned} \tag{1.27}$$

it follows

$$D_4 = F_2 = E_3 = 0. \tag{1.28}$$

The rest of the coefficients can be easily determined from the formulas (1.14), (1.16), (1.18), (1.20). Substituting back these coefficients in (1.9) one gets (1.8). **q.e.d.**

The Riemannian curvature 2-form $\Omega$ is defined by

$$d\theta_q{}^p - \theta_q{}^s \wedge \theta_s{}^p = \frac{1}{2}\breve{R}_q{}^p{}_{rs}\theta^r \wedge \theta^s, \tag{1.29}$$

where $\breve{R}_q{}^p{}_{rs}$ are the Riemannian curvature coefficients of the Riemannian metric $\breve{a}$ with respect to the orthonormal coframe $\{\theta^1, \theta^2, \theta^3, \theta^4\}$. We notice that since $\breve{a}_{ij} = \delta_{ij}$, the coefficients $\breve{R}_q{}^p{}_{rs}$ are numerically equal to $\breve{R}_{qprs}$.

Using (1.8) and (1.29) we obtain by direct calculation:

**Proposition 1.2** *The Riemannian curvature coefficients of the metric* $\breve{a}$ *with respect to the orthonormal coframe* $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ *are:*

$$\begin{array}{llllll}
\breve{R}_{1212} = A & \breve{R}_{1213} = 0 & \breve{R}_{1214} = 0 & \breve{R}_{1223} = 0 & \breve{R}_{1224} = 0 & \breve{R}_{1234} = -E \\
\breve{R}_{1312} = 0 & \breve{R}_{1313} = A & \breve{R}_{1314} = 0 & \breve{R}_{1323} = 0 & \breve{R}_{1324} = E & \breve{R}_{1334} = 0 \\
\breve{R}_{1412} = 0 & \breve{R}_{1413} = 0 & \breve{R}_{1414} = B & \breve{R}_{1423} = 2E & \breve{R}_{1424} = 0 & \breve{R}_{1434} = 0 \\
\breve{R}_{2312} = 0 & \breve{R}_{2313} = 0 & \breve{R}_{2314} = 2E & \breve{R}_{2323} = C & \breve{R}_{2324} = 0 & \breve{R}_{2334} = 0 \\
\breve{R}_{2412} = 0 & \breve{R}_{2413} = E & \breve{R}_{2414} = 0 & \breve{R}_{2423} = 0 & \breve{R}_{2424} = D & \breve{R}_{2434} = 0 \\
\breve{R}_{3412} = -E & \breve{R}_{3413} = 0 & \breve{R}_{3414} = 0 & \breve{R}_{3423} = 0 & \breve{R}_{3424} = 0 & \breve{R}_{3434} = D.
\end{array}$$

*where*

$$\begin{aligned}
A &= \frac{\mu'\varphi'}{\mu\varphi} - \frac{\mu^2}{\varphi^4} \\
B &= \frac{\mu''}{\mu} \\
C &= \frac{3\mu^2}{\varphi^4} + \frac{(\varphi')^2 - 4}{\varphi^2} \\
D &= \frac{\varphi''}{\varphi} \\
E &= \frac{\mu\varphi'}{\varphi^3} - \frac{\mu'}{\varphi^2}
\end{aligned}$$

**Proposition 1.3** *Taking into account that* $\varphi = \sin r$ *and* $\mu = \sin r \cos r$, *the Riemann curvature coefficients become:*

$$\begin{array}{llllll}
\breve{R}_{1212} = -1 & \breve{R}_{1213} = 0 & \breve{R}_{1214} = 0 & \breve{R}_{1223} = 0 & \breve{R}_{1224} = 0 & \breve{R}_{1234} = -1 \\
\breve{R}_{1312} = 0 & \breve{R}_{1313} = -1 & \breve{R}_{1314} = 0 & \breve{R}_{1323} = 0 & \breve{R}_{1324} = 1 & \breve{R}_{1334} = 0 \\
\breve{R}_{1412} = 0 & \breve{R}_{1413} = 0 & \breve{R}_{1414} = -4 & \breve{R}_{1423} = 2 & \breve{R}_{1424} = 0 & \breve{R}_{1434} = 0 \\
\breve{R}_{2312} = 0 & \breve{R}_{2313} = 0 & \breve{R}_{2314} = 2 & \breve{R}_{2323} = -4 & \breve{R}_{2324} = 0 & \breve{R}_{2334} = 0 \\
\breve{R}_{2412} = 0 & \breve{R}_{2413} = 1 & \breve{R}_{2414} = 0 & \breve{R}_{2423} = 0 & \breve{R}_{2424} = -1 & \breve{R}_{2434} = 0 \\
\breve{R}_{3412} = -1 & \breve{R}_{3413} = 0 & \breve{R}_{3414} = 0 & \breve{R}_{3423} = 0 & \breve{R}_{3424} = 0 & \breve{R}_{3434} = -1.
\end{array}$$

We recall that the Ricci tensor of a Riemannian manifold is given by

$$\breve{Ric}_{ij} = \breve{R}_{i\ sj}^{\ s}. \tag{1.30}$$

A Riemannian manifold $(M, \breve{a})$ is called Einstein space if

$$\breve{Ric}_{ij} = \breve{Ric}(x)\breve{a}_{ij}. \tag{1.31}$$

From Proposition 1.3 and (1.31), a simple calculation gives

**Theorem 1.1** *The complex projective space* $\mathbb{CP}^2$ *endowed with the Riemannian metric* $\breve{a}$ *from (1.3) is an Einstein space with the constant* $\breve{Ric} = 6$.

In the following we consider the sectional curvature:

$$sec(v, w) = -\frac{\breve{R}(v, w, v, w)}{\breve{a}(v, v)\breve{a}(w, w) - [\breve{a}(v, w)]^2}. \tag{1.32}$$

Since $\{F_1, F_2, F_3, F_4\}$ is the dual orthonormal frame of the coframe $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ with respect to $\breve{a}$, we obtain that

$$sec(F_i, F_j) = \breve{a}(\Re(F_i \wedge F_j), F_i \wedge F_j),$$

where $\Re : \Lambda^2(\mathbb{CP}^2) \longrightarrow \Lambda^2(\mathbb{CP}^2)$, $\breve{a}(\Re(X \wedge Y), V \wedge W) = \breve{R}(X \wedge Y, V \wedge W)$ is the *curvature operator*, see [1].

Consequently, we have

$$\breve{a}(\Re(F_i \wedge F_j), F_k \wedge F_l) = \breve{R}(F_i \wedge F_j, F_k \wedge F_l) = \breve{R}(F_i, F_j, F_l, F_k) = \breve{R}_{ijlk} \tag{1.33}$$

and from Proposition 1.3 we obtain that

$$\begin{array}{ll} sec(F_1, F_2) = -\breve{R}_{1212} = 1 & sec(F_2, F_3) = -\breve{R}_{2323} = 4 \\ sec(F_1, F_3) = -\breve{R}_{1313} = 1 & sec(F_3, F_4) = -\breve{R}_{3434} = 1 \\ sec(F_1, F_4) = -\breve{R}_{1414} = 4 & sec(F_2, F_4) = -\breve{R}_{2424} = 1. \end{array}$$

We can observe that all sectional curvatures lie in the interval $[1, 4]$.

At the same time, we can write the matrix of the operator $\Re$ in the basis

$$\frac{1}{\sqrt{2}}(F_1 \wedge F_2 \pm F_3 \wedge F_4), \frac{1}{\sqrt{2}}(F_1 \wedge F_3 \pm F_4 \wedge F_2), \frac{1}{\sqrt{2}}(F_1 \wedge F_4 \pm F_2 \wedge F_3),$$

that is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are $6,\ 0,\ 0,\ 2,\ 2,\ 2$.

**Proposition 1.4** *The complex projective space* $\mathbb{CP}^2$ *is with parallel curvature tensor, i.e.* $\mathbb{CP}^2$ *is a locally symmetric space.*

Proof. As we know, the covariant derivative has the poperty:

$$(\breve{\nabla}_Z \Re)(X \wedge Y) = \breve{\nabla}_Z(\Re(X \wedge Y)) - \Re(\breve{\nabla}_Z(X \wedge Y))$$

Since the eigenvalues of the curvature operator are constant, i.e. $\Re(\omega) = \lambda\omega$, $\lambda$ being constant, we obtain that $\breve{\nabla}\Re = 0$. **q.e.d.**

# 2 Finsler Geometry of $\mathbb{CP}^2$

In [4] it is studied the following generalization of Zermelo's problem of navigation to Finsler spaces. Consider a manifold $M$ with a Riemannian metric $\breve{\alpha} = \sqrt{\breve{a}_{ij}(x)y^i y^j}$. If a ball rolls about $M$ with constant speed 1, then any geodesic is a path of shortest time. Next, suppose a *wind* $W = W^i \dfrac{\partial}{\partial x^i}$, such that $\breve{\alpha}(W) < 1$, blows over $M$ (as an external force acting on the ball).

It was shown [4], that any path of the shortest time for the ball is a geodesic of the Randers metric $F = \alpha + \beta$, where

$$\tilde{a}_{ij} = \frac{\breve{a}_{ij}[1 - \breve{\alpha}^2(W)] + \breve{W}^i \breve{W}^j}{[1 - \breve{\alpha}^2(W)]^2}, \qquad \tilde{b}_i = \frac{-\breve{W}^i}{1 - \breve{\alpha}(W)}, \tag{2.1}$$

where $\breve{W}_i = \breve{a}_{ij} W^j$. The assumption $\breve{\alpha}(W) < 1$ implies $||b|| < 1$, therefore $F$ is a strongly convex Finsler metric.

The inverse problem is also considered. Namely, can every Randers metric $F = \alpha + \beta$ be realized as a solution to Zermelo's problem of navigation on a Riemannian manifold $(M, \breve{a})$ under an external force $W$? The answer is affirmative, [4].

Let us recall that a Finsler space $(M, F)$ is an Einstein space if the Ricci curvature is a function of $x \in M$ alone. Namely, let us denote the Ricci scalar of $F$ by

$$Ric\ (x, y) := \frac{1}{F^2} K^s{}_s \tag{2.2}$$

where $K^s{}_s$ is the spray curvature.

The *Ricci tensor* of a Finsler metric $F$ is then

$$Ric_{ij} := \frac{1}{2} \frac{\partial^2 [F^2 Ric(x, y)]}{\partial y^i \partial y^j} \tag{2.3}$$

It is known that a Finsler metric is Einstein if

$$Ric_{ij} = Ric\ (x) g_{ij} \tag{2.4}$$

where $g_{ij}$ is the fundamental tensor of $(M, F)$.

Robles [9] proved the following result:

**Theorem 2.1** *Suppose the Randers metric $F = \alpha + \beta$ solves Zermelo's problem of navigation on the Riemannian manifold $(M, \breve{a})$ under the external force $W$, $\breve{\alpha}(W) < 1$. Then $(M, F)$ is an Einstein space with Ricci scalar $Ricc := (n-1)K$ if and only if*

*(1) the Riemannian metric $\breve{a}$ is Einstein with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$,*

*(2) the vector field $W$ is an infinitesimal homothety of $\breve{a}$, i.e.*

$$\breve{W}_{i:j} + \breve{W}_{j:i} = -\sigma \breve{a}_{ij} \tag{2.5}$$

*is satisfied, where ":" denotes the covariant derivative of the Levi-Civita connection of the Riemannian metric $\breve{a}$.*

Now, let us consider our question:

"There are Einstein metrics of Randers type on $\mathbb{CP}^2$?" If we can construct $\breve{a}$ and $W$ that satisfy (1), (2) in Theorem 2.1, then the answer is affirmative.

Theorem 1.3 states that $(\mathbb{CP}^2, \breve{a})$, with $\breve{a}$ in (1.3) is an Einstein metric with the Riemannian Ricci scalar $\breve{Ric} = 6$. In other words (1) is satisfied and $K + \frac{1}{16}\sigma^2 = 2$.

Let us consider the external force on the form

$$W = F_1 W^1(r) + F_2 W^2(r) + F_3 W^3(r) + F_4 W^4(r). \tag{2.6}$$

Since $\breve{a}_{ij} = \delta_{ij}$, if $\breve{W}_i(r) = \breve{a}_{ij} W^j(r)$ then, numerically $\breve{W}^i$ and $\breve{W}_i$ are equal for any $i \in \{1, 2, 3, 4\}$. The contravariant derivative with respect to the Riemannian metric $\breve{a}$ can be calculated with the formula

$$\breve{W}_{i:j} = \left( d\breve{W}_i - \breve{W}_s \theta_i^{\ s} \right) (F_j) \tag{2.7}$$

where $\theta_i^{\ s}$ are the Levi-Civita connection forms calculated in (1.8). Straightforward calculations give:

(2.8)

$$\begin{pmatrix} \breve{W}_{1:1} & \breve{W}_{1:2} & \breve{W}_{1:3} & \breve{W}_{1:4} \\ \breve{W}_{2:1} & \breve{W}_{2:2} & \breve{W}_{2:3} & \breve{W}_{2:4} \\ \breve{W}_{3:1} & \breve{W}_{3:2} & \breve{W}_{3:3} & \breve{W}_{3:4} \\ \breve{W}_{4:1} & \breve{W}_{4:2} & \breve{W}_{4:3} & \breve{W}_{4:4} \end{pmatrix} = \begin{pmatrix} \breve{W}_4 \dfrac{\mu'}{\mu} & -\breve{W}_3 \dfrac{\mu}{\varphi^2} & \breve{W}_2 \dfrac{\mu}{\varphi^2} & \breve{W}_1' \\ -\breve{W}_3 (\dfrac{\mu}{\varphi^2} - \dfrac{2}{\mu}) & \breve{W}_4 \dfrac{\varphi'}{\varphi} & -\breve{W}_1 \dfrac{\mu}{\varphi^2} & \breve{W}_2' \\ -\breve{W}_2 (\dfrac{2}{\mu} - \dfrac{\mu}{\varphi^2}) & \breve{W}_1 \dfrac{\mu}{\varphi^2} & \breve{W}_4 \dfrac{\varphi'}{\varphi} & \breve{W}_3' \\ -\breve{W}_1 \dfrac{\mu'}{\mu} & -\breve{W}_2 \dfrac{\varphi'}{\varphi} & -\breve{W}_3 \dfrac{\varphi'}{\varphi} & \breve{W}_4' \end{pmatrix}$$

and (2.5) implies

$$\begin{cases} 4\breve{W}_4 \cot(2r) = -\sigma \\ \breve{W}_3 \tan(r) = 0 \\ \breve{W}_2 \tan(r) = 0 \\ \breve{W}_1' - 2\breve{W}_1 \cot(2r) = 0 \\ 2\breve{W}_4 \cot(r) = -\sigma \\ \breve{W}_4 \cot(2r) = -\sigma \\ \breve{W}_2' - \breve{W}_2 \cot(r) = 0 \\ 2\breve{W}_4 \cot(r) = -\sigma \\ \breve{W}_3' - \breve{W}_3 \cot(r) = 0 \\ \breve{W}_4' = -\sigma \end{cases} \tag{2.9}$$

where $r \in [0, \dfrac{\pi}{2}]$.

A close look at the equations (2.9) shows that, for $r \in (0, \dfrac{\pi,}{2})$, we get $\breve{W}_2 = 0,\ \breve{W}_3 = 0,\ \breve{W}_4 = 0,\ \sigma = 0.$

As for $\breve{W}_1$, it satisfies the differential equation

$$\breve{W}_1' - 2\breve{W}_1 \cot(2r) = 0.$$

This is an ordinary differential equation with separable variables and it gives the solution

$$\breve{W}_1(r) = \lambda \sin(2r), \qquad \lambda \neq 0. \tag{2.10}$$

If we put the condition $\alpha^2(W) < 1$ it follows $\lambda \in (-1,1) \setminus \{0\}$. Therefore we have found a Killing vector

$$W = F_1 W^1 = F_1 \lambda \sin(2r), \quad \lambda \in (-1,1) \setminus \{0\}. \tag{2.11}$$

that satisfies condition (2.5). Then, from Theorem 2.1 it follows that the Randers space $F = \alpha + \beta$, with $\alpha^2 = \tilde{a}_{ij} y^i y^j, \quad \beta = \tilde{b}_i y^i$, where

(2.12)
$$\tilde{a}_{ij} = \begin{pmatrix} \dfrac{1}{[1-\lambda^2 \sin^2(2r)]^2} & 0 & 0 & 0 \\ 0 & \dfrac{1}{1-\lambda^2 \sin^2(2r)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{1-\lambda^2 \sin^2(2r)} & 0 \\ 0 & 0 & 0 & \dfrac{1}{1-\lambda^2 \sin^2(2r)} \end{pmatrix}$$

and

$$\tilde{b}_1 = -\frac{\lambda \sin(2r)}{1-\lambda^2 \sin^2(2r)}, \qquad \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4 = 0, \tag{2.13}$$

is an Einstein metric of Randers type.

Therefore we have

**Theorem 2.2** *The Randers space $(\mathbb{CP}^2, \alpha + \beta)$, where $\alpha$ and $\beta$ are given by (2.12) and (2.13) is an Einstein-Finsler metric with the Ricci constant curvature $Ricc = 6$.*

Next, we would like to compare the Riemannian geometry of this Einstein metric of Raders type with the Riemannian geometry of $\mathbb{CP}^2$.

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