The L-dual of a Matsumoto Space

Ioana Monica MASCA, Vasile Sorin SABAU, Hideo SHIMADA

June 22, 2006

Abstract

In this paper we shall give the L-dual of a Matsumoto space. In [3, 8] the L-duals of a Randers and Kropina space were introduced.

Keywords: Matsumoto space, Finsler space, Cartan space, the duality between Finler and Cartan spaces.

1 Introduction

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β)-metric if F is homogeneous function of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^iy^j$, $y = y^i\frac{\partial}{\partial x^i} \mid_x \in T_xM$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\tilde{TM} = TM \setminus \{0\}$.
A Finsler space with the fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y) \tag{1}$$

is called a Randers space.
A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)} \tag{2}$$

is called a Kropina space and one with

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)} \tag{3}$$

is called a Matsumoto space.
Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x, p)$. We also consider Cartan spaces having the metric function of the following forms:

$$K(x, p) = \sqrt{a^{ij}(x)p_ip_j} + b^i(x)p_i \tag{4}$$

or

$$K(x,p) = \frac{a^{ij}(x)p_i p_j}{b^i(x)p_i} \tag{5}$$

with $a_{ij}a^{jk} = \delta_i^k$ and we will again call these spaces Randers and, respectively, Kropina spaces on the cotangent bundle T^*M.
Let $L(x,y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x,p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$.

As we know [8] if L is a diferentiable map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$:

$$\varphi(x,y) = (x^i, \dot{\partial}_a L(x,y)) \tag{6}$$

which is called the Legendre transformation. We can define, in this case, the function $H : U^* \longmapsto R$:

$$H(x,p) = p_a y^a - L(x,y), \tag{7}$$

where $y = (y^a)$ is the solution of the equations:

$$p_a = \dot{\partial}_a L(x,y). \tag{8}$$

In the same manner, the fiber derivative is given locally by:

$$\psi(x,p) = (x^i, \dot{\partial}^a H(x,p)), \tag{9}$$

ψ is a diffeomorphism between the same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \longmapsto R$:

$$L(x,y) = p_a y^a - H(x,p), \tag{10}$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}^a H(x,p). \tag{11}$$

The Hamiltonian given by (7) is called the Legendre transformation of the Lagrangian L and the Lagrangian given by (10) is called the Legendre transformation of the Hamiltonian H.

If (M,K) is a Cartan space, then (M,H) is a Hamilton manifold [8] where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogenous on a domain of T^*M. So, we get the following transformation of H on U:

$$L(x,y) = p_a y^a - H(x,p) = H(x,p). \tag{12}$$

Proposition 1.1 [8] The scalar field $L(x,y)$ definited by (12) is a positively 2-homogeneous regular Lagrangian on U.
Therefore, we get Finsler metric F of U, so that

$$L = \frac{1}{2}F^2 \tag{13}$$

Thus, for the Cartan space (M, F) we always can locally associate a Finsler space (M, F) which will be called the L-dual of a Cartan space $(M, C_{|U^*})$ Vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the L-dual of a Finsler space $(M, F_{|U})$.

2 The (α, β) Finsler - (α^*, β^*) Cartan L-duality

Theorem 2.1 [3, 8] Let (M, F) be a Randers space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian lengh of b_i. Then:

1. If $b^2 = 1$, the L-dual of (M, F) is a Kropina space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\frac{a^{ij}p_i p_j}{2b^i p_i}\right)^2. \tag{14}$$

2. If $b^2 \neq 1$, the L-dual of (M, F) is a Randers space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\sqrt{\tilde{a}^{ij}p_i p_j} \pm \tilde{b}^i p_i\right)^2, \tag{15}$$

where

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^i b^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i$$

(in (15) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

Theoreme 2.2 [3, 8] The L-dual of a Kropina space is a Randers space on T^*M with the Hamiltonian:

$$H(x,p) = \frac{1}{2}\left(\sqrt{\tilde{a}^{ij}p_i p_j} \pm \tilde{b}^i p_i\right)^2, \tag{16}$$

where

$$\tilde{a}^{ij} = \frac{b^2}{4}a^{ij}; \quad \tilde{b}^i = \frac{1}{2}b^i$$

(in (16) $'-'$ corresponds to $\beta < 0$ and $'+'$ corresponds to $\beta > 0$).

In [3] $\alpha^* = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$, $\beta^* = b^i(x)p_i$ where $a^{ij}(x)$ are the reciprocal components of a_{ij} and $b^i(x)$ are the components of the vector field on M, $b^i(x) = a^{ij}(x)b_j(x)$. We can consider the metric functions $K = \alpha^* + \beta^*$ (Randers metric on T^*M) or $K = \frac{\alpha^{*2}}{\beta^*}$ (Kropina metric on T^*M) defined on a domain $D^* \subset T^*M$. So, we can easyly rewrite the previous theorems:

Theorem 2.1' Let (M, F) be a Randers space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian lengh of b_i. Then:

1. If $b^2 = 1$, the L-dual of (M, F) is a Kropina space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\frac{\alpha^{*2}}{2\beta^*}\right)^2. \tag{17}$$

2. If $b^2 \neq 1$, the L-dual of (M, F) is a Randers space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{18}$$

   with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_i p_j}$ where

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^i b^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i$$

   (in (18) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

Theorem 2.2' The L-dual of a Kropina space is a Randers space on T^*M with the Hamiltonian:

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{19}$$

where

$$\tilde{a}^{ij} = \frac{b^2}{4}a^{ij}; \quad \tilde{b}^i = \frac{1}{2}b^i$$

(in (19) $'-'$ corresponds to $\beta < 0$ and $'+'$ corresponds to $\beta > 0$).

Theorem 2.3 Let (M, F) be a Matsumoto space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian lengh of b_i. Then

1. If $b^2 = 1$, the L-dual of (M, F) is the space having the fundamental function:

$$H(x,p) = \frac{1}{2}\left(-\frac{b^i p_i}{2}\frac{\left(\sqrt[3]{p_i p_j a^{ij}} + \sqrt[3]{\left(p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}\right)^2}\right)^3}{p_i p_j a^{ij} + \left(p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}\right)^2}\right)^2, \tag{20}$$

   where

$$\tilde{a}^{ij} \quad = \quad b^i b^j - a^{ij}.$$

2. If $b^2 \neq 1$, the L-dual of (M, F) is the space on T^*M having the fundamental function:

$$H(x,p) = \frac{1}{2}\left(-\frac{b^i p_i}{200}\frac{25\left(2\sqrt{p_i p_j d_2^{ij}} + \sqrt{p_i p_j d_4^{ij}}\right)^2 + p_i p_j d_8^{ij}}{\sqrt{p_i p_j d_2^{ij}}\sqrt{p_i p_j d_4^{ij}} + p_i p_j d_9^{ij}}\right)^2, \tag{21}$$

where

$$\begin{aligned}
c_1^{ij} &= (b^i b^j + 2\varepsilon_1 a^{ij})^2 + (2a^{ij})^2\varepsilon_3,\\
c_2^{ij} &= a^{ij}(\theta_4^2 b^i b^j + a^{ij}\varepsilon_2),\\
c_3^{ij} &= (2a^{ij})^2\theta_5^3,\\
\sqrt[3]{\tilde{a}^{ij\,2}} &= \sqrt[3]{c_1^{ij}} - 2\sqrt[3]{c_2^{ij}} + \sqrt[3]{c_3^{ij}},\\
d_1^{ij} &= d_3^{ij} + 4m(a^{ij}b^2 - b^i b^j),\\
d_2^{ij} &= \sqrt{d_3^{ij}a^{ij}} + 4\sqrt{d_1^{ij}a^{ij}} - d_3^{ij},\\
d_3^{ij} &= 2\sqrt[3]{2a^{ij}(\tilde{a}^{ij})^2},\\
\sqrt{d_4^{ij}} &= \sqrt{d_3^{ij}} + 3\sqrt{a^{ij}},\\
\sqrt{d_5^{ij}} &= \sqrt{d_3^{ij}a^{ij}},\\
d_6^{ij} &= d_1^{ij}a^{ij},\\
\sqrt{d_7^{ij}} &= 2\sqrt{d_2^{ij}} + \sqrt{d_4^{ij}},\\
d_8^{ij} &= 200\Big(\sqrt{d_6^{ij}} + 2na^{ij}\Big) - 5\Big(4\sqrt{d_3^{ij}} + \sqrt{d_4^{ij}}\Big),\\
d_9^{ij} &= 4\sqrt{d_6^{ij}} + 4a^{ij}p + 9\sqrt{d_5^{ij}},
\end{aligned}$$

and

$$\begin{aligned}
m &= 1 - b^2,\\
n &= \frac{20b^2 - 29}{29},\\
p &= \frac{1 - 2b^2}{2},\\
\theta_1 &= -\frac{712b^6 - 452b^4 + 24b^2 + 1}{1728},\\
\theta_2 &= \frac{576b^4 - 2232b^2 + 2628}{1728},\\
\theta_3 &= -\left(\frac{8b^2 + 1}{12}\right)^2,\\
\theta_4 &= \frac{2b^2 + 1}{6},\\
\theta_5 &= \frac{11b^2 + 1}{12},\\
\varepsilon_1 &= 2(\theta_4^2 - \theta_2),\\
\varepsilon_2 &= 3\theta_3\theta_4^2 + \theta_2^2,
\end{aligned}$$

$$\varepsilon_3 = 4\varepsilon_2 - 2\theta_1 - \varepsilon_1.$$

Proof: We put:$\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$. We have $F = \frac{\alpha^2}{\alpha - \beta}$, and

$$p_i = \frac{1}{2}\dot{\partial}_i F^2 = \frac{y_i}{\alpha - \beta} + \frac{\alpha^2 b^i - y_i \beta}{(\alpha - \beta)^2}. \tag{22}$$

Contracting in (22) by p^i and b^i we get:

$$\alpha^{*2} = \frac{F}{(\alpha - \beta)^2}[F^2(\alpha - 2\beta) + \alpha^2 \beta^*] \tag{23}$$
$$\beta^* = \frac{F}{(\alpha - \beta)^2}[\beta(\alpha - 2\beta) + \alpha^2 b^2].$$

In [10], for a Finsler (α, β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0$, $\forall x \in M$.
ϕ satisfies $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi"(s) > 0$, $(|s| \leq b_0)$.
A Matsumoto metric is a special (α, β)-metric with $\phi = \frac{1}{1-s}$.
Using Shen's [11] notation $s = \frac{\beta}{\alpha}$ (23) become:

$$\alpha^{\star 2} = F^2 \frac{1 - 2s}{(1 - s)^3} + F \frac{1}{(1 - s)^2} \beta^\star \tag{24}$$
$$\beta^\star = Fs \frac{1 - 2s}{(1 - s)^2} + F \frac{1}{(1 - s)^2} b^2.$$

Now we put $1 - s = t$, so $s = 1 - t$ and both equations become:

$$\alpha^{\star 2} = F^2 \frac{2t - 1}{t^3} + F \frac{1}{t^2} \beta^\star \tag{25}$$
$$\beta^\star = F(1 - t)\frac{2t - 1}{t^2} + F \frac{1}{t^2} b^2. \tag{26}$$

We get

$$\beta^* t^2 = M(-2t^2 + 3t + b^2 - 1) \tag{27}$$

For $b^2 = 1$ from (26) we get

$$F = -\frac{\beta^* t}{2t - 3} \tag{28}$$

and substitute F in (25), after computation we have a cubic equatin:

$$t^3 - 3t + \frac{9}{4}t - \frac{\beta^\star}{2\alpha^{\star 2}} = 0. \tag{29}$$

Using Cardano's method for solving cubic equation [11], we get:

$$F = -\frac{\beta^\star}{2}\frac{(2P-1)^2}{3P^2+(P-1)^2}, \tag{30}$$

where for P we have:

$$P = \frac{1}{2}\sqrt[3]{\left(\frac{\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}}}{\alpha^\star}\right)^2}. \tag{31}$$

After computation, for F we get:

$$F = -\frac{\beta^\star}{2}\frac{\left(\sqrt[3]{\alpha^{\star 2}} + \sqrt[3]{(\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}})^2}\right)^3}{\alpha^{\star 2} + (\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}})^2}. \tag{32}$$

Replacing now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$ we can easyly get (20).

If $b^2 \neq 1$ (28) is more complicated because:

$$F = \frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}, \tag{33}$$

and puting this in (25) we get the quadric equation:

$$t^4 - 3t^3 + t^2\frac{13-4b^2}{4} + t\frac{6\alpha^{*2}(b^2-1)}{4\alpha^{*2}} + \frac{\alpha^{*2}(b^2-1)^2 + \beta^{*2}(1-b^2)}{4\alpha^{*2}} = 0. \tag{34}$$

After laborious calculs, [11], (34) becomes a cubic equation (different from (29))and solving this we get:

$$\begin{aligned} F &= -\frac{\beta^*}{2}\left(\left(\left(\sqrt{-A^2+3A+2\sqrt{A^2+m\left(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\right)}} + \frac{A}{2}+\frac{3}{4}\right)^2\right.\right. \\ &+ \left.\sqrt{A^2+m\left(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\right)} - \frac{5}{4}\left(A+\frac{3}{10}\right)^2 + n\right)/ \\ &/ \left(\left(\frac{3}{2}+2A\right)(\sqrt{-A^2+3A+2\sqrt{A^2+m\left(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\right)}}\right. \\ &+ \left. 2\sqrt{A^2+m\left(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\right)} + \frac{9}{2}A + p\right), \end{aligned} \tag{35}$$

where

$$A^2 = \sqrt[3]{\left(\frac{1}{2}\frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_1\right)^2 + \varepsilon_3} + \sqrt[3]{-4\left(\theta_4^3\frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_2\right)} + \theta_5. \tag{36}$$

Replacing now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, after computation, we get for (36), (21).
It ie easy to see that both relations, (20) and (21), are comming from (27). Replacing $b^2 = 1$ we get the cubic equatoin (29). As solution, we find (20). For $b^2 \neq 1$, in (27) we get the complicated quadric equation (34) with (21) as solution. If in (34) we replace $b^2 = 1$ we get $t^4 - 3t^3 + \frac{9}{4} = 0$ with $t_1 = t_2 = 0$ and $t_3 = t_4 = \frac{3}{2}$. It is impossible for these four solutions to exist in our proof. So, we can easyly see that (20) and (21) are two different relations and we can't get (20) as a particular case of (21).

Remarks

1. Using α^* and β^* we can get, for the L-dual of (M, F), if $b^2 = 1$, the fundamental function:

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^\star}{2} \frac{\left(\sqrt[3]{\alpha^{\star 2}} + \sqrt[3]{\left(\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}}\right)^2} \right)^3}{\alpha^{\star 2} + (\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}})^2} \right)^2. \tag{37}$$

2. In (20) $\tilde{a}^{ij}$ is positive-definite and the Randers metric on T^*M $p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}$ is positive-valued for any p.

3 Bibliography

[1] Antonelly,P.L., Ingarden,R.S.,Matsumoto.M., "The Theory of Sprays and Finsler Spaces with Aplications in Physics and Biology",Kluwer Acad.Publ.FTPH no 58,1993.
[2] Bao,D., Chern,S.S., Shen,Z., "An introduction to Riemannian-Finsler Geometry (Graduate Texts in Mathematics;200),Springer-Verlag,2000.
[3] Hrimiuc,D., Shimada,H., "On the L-duality between Lagrange and Hamilton manifolds",Nonlinear World,3(1996),613-641.
[4] Hrimiuc,D., Shimada,H., "On some special problems concerning the L-duality between Finsler and Cartan spaces", Tensor N.S.,58(1997),48-61.
[5] Matsumoto,M., "Foundations of Finsler Geometry and Spaces",Kaisheisha Press, Otsu,1986.
[6] Miron,R.,"Cartan Spaces in a new point of view by considering them as duals of Finsler Spaces", Tensor N.S.46(1987),330-334.
[7] Miron,R.,Anastasiei,M.,"The Geometry of Lagrange Spaces: Theory and Applications", Kluwer Acad.Publ.FTPH no 59(1994)
[8] Miron,R., Hrimiuc,D., Shimada,H.,Sabau,S.V., "The Geometry of Hamilton and Lagrange Spaces", Kluwer Acad.Publ.FTPH no 118(2001).
[9] Sabau,S.V., Shimada,H.,"Classes of Finsler spaces with (α, β)-metrics", Rep.on Mathematical Physics 47(2001),31-48.
[10] Shen,Z., "On Landsberg (α, β)-metrics",http://www.math.iupiu.edu/ $\sim$

zshen/ Research /papers,2006.
[11] http:// en.wikipedia.org/wiki/.